PATTERNS IN RECTANGULATIONS: $\top$-LIKE PATTERNS, INVERSION SEQUENCES, AND DYCK PATHS

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joint work with Andrei Asinowski[2] (Alpen-Adria-Universität Klagenfurt)

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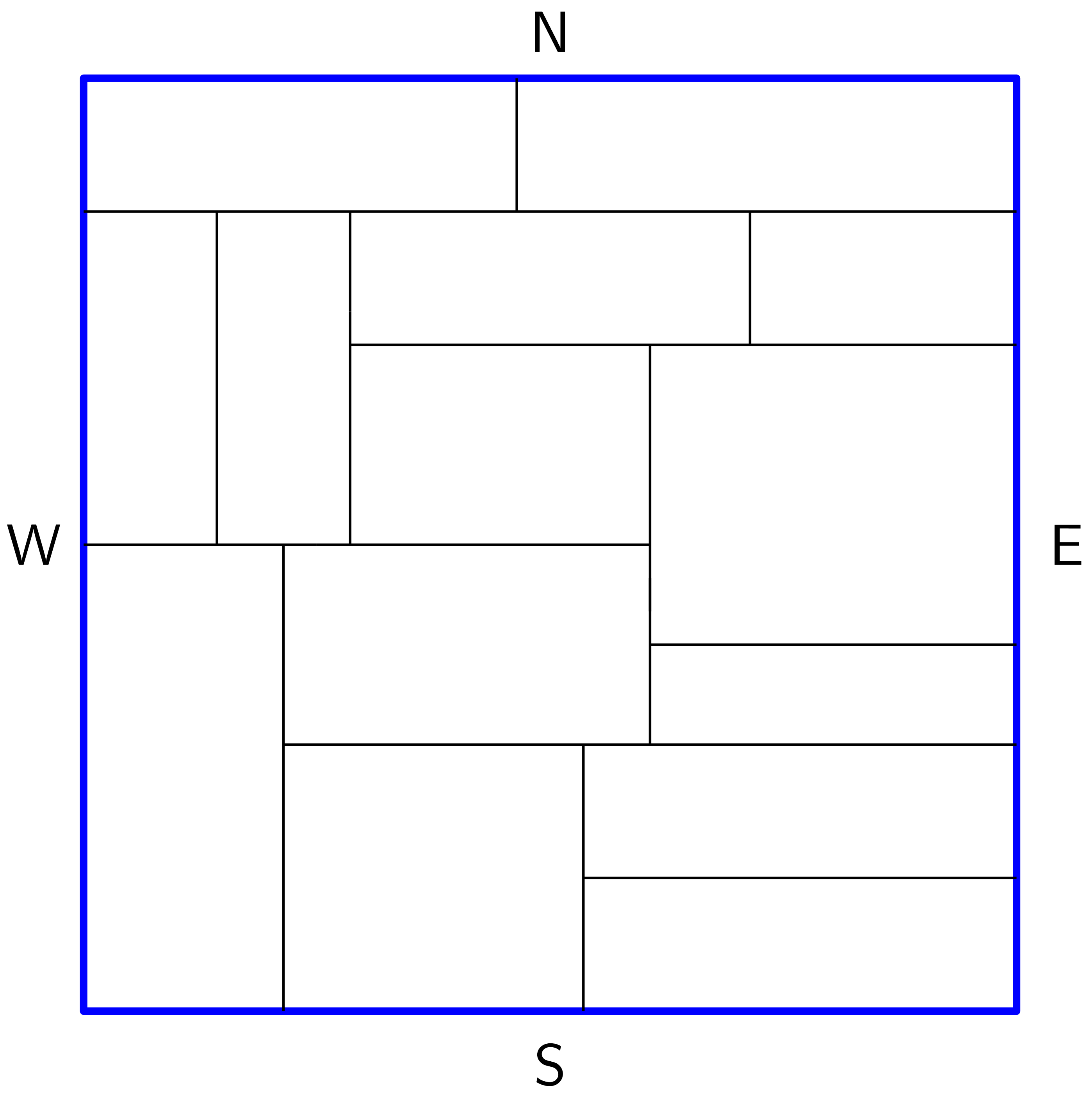
[2] Supported by FWF – Austrian Science Fund

Definitions and Terminology

A *rectangulation* $\mathcal{R}$ is a partition of a rectangle R into a finite number of rectangles such that there are no $+$ joints.



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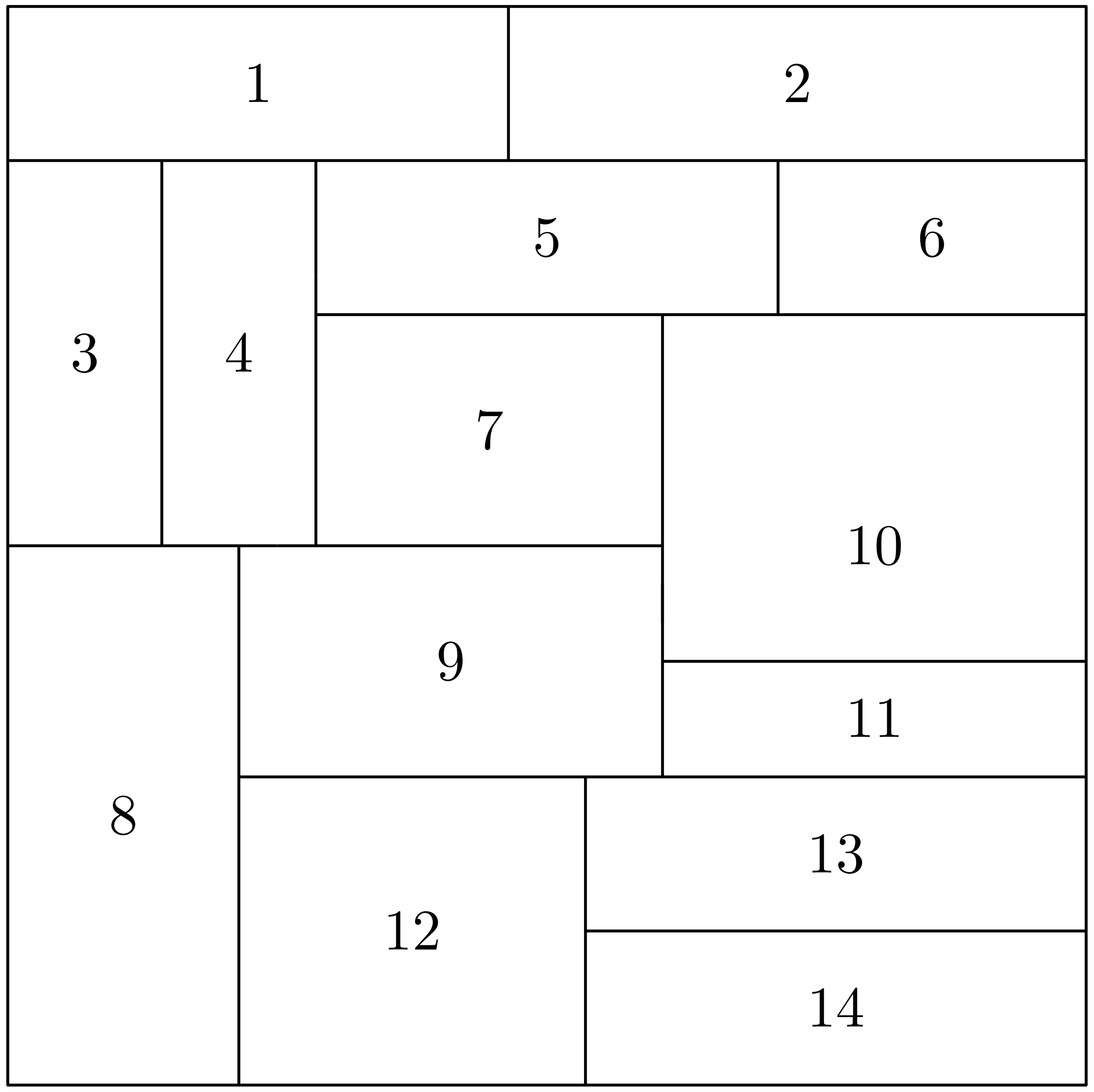

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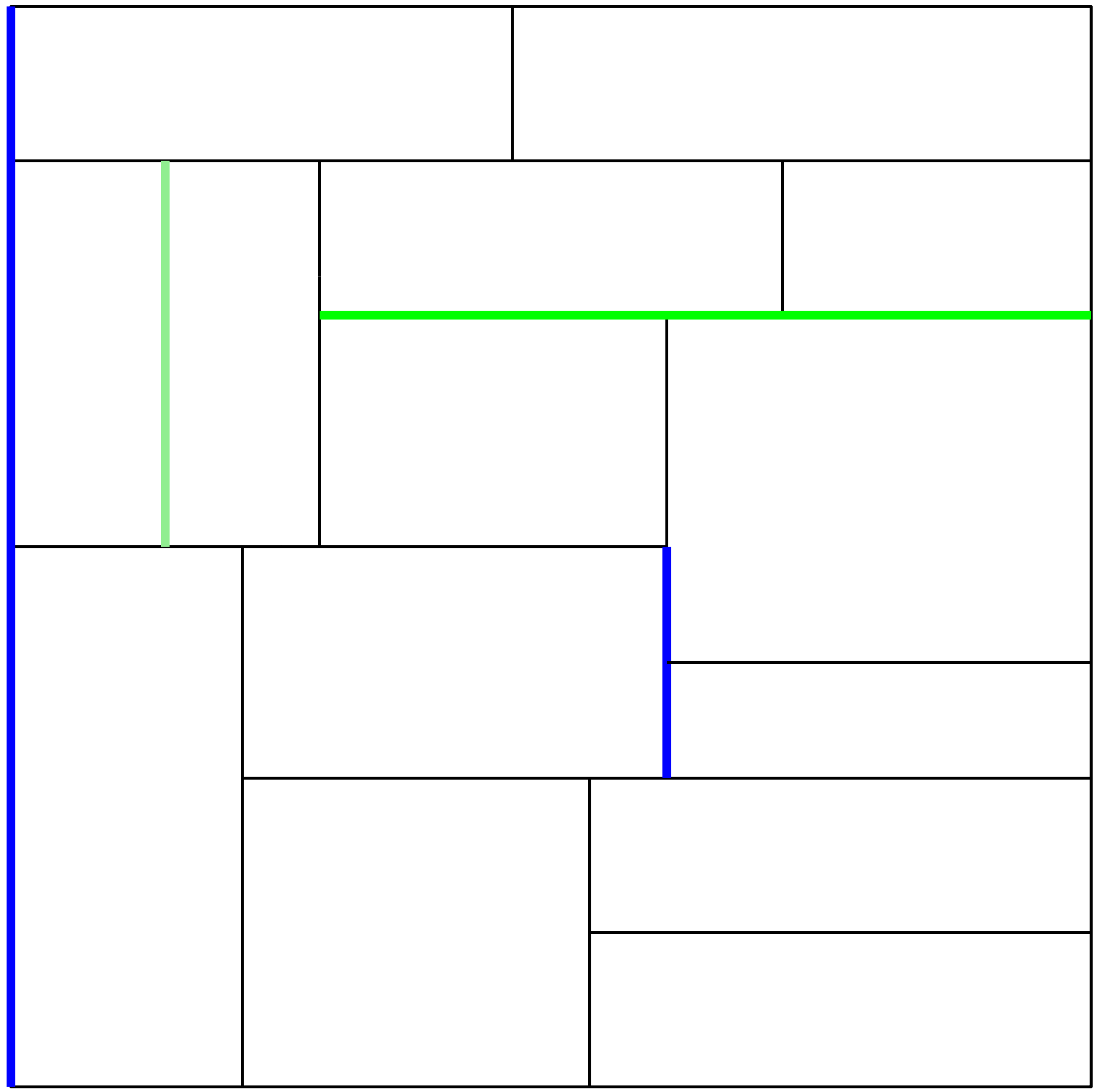
A *segment* in $\mathcal{R}$ is a maximal union of rectangle edges which form a straight line (and not an edge of R).



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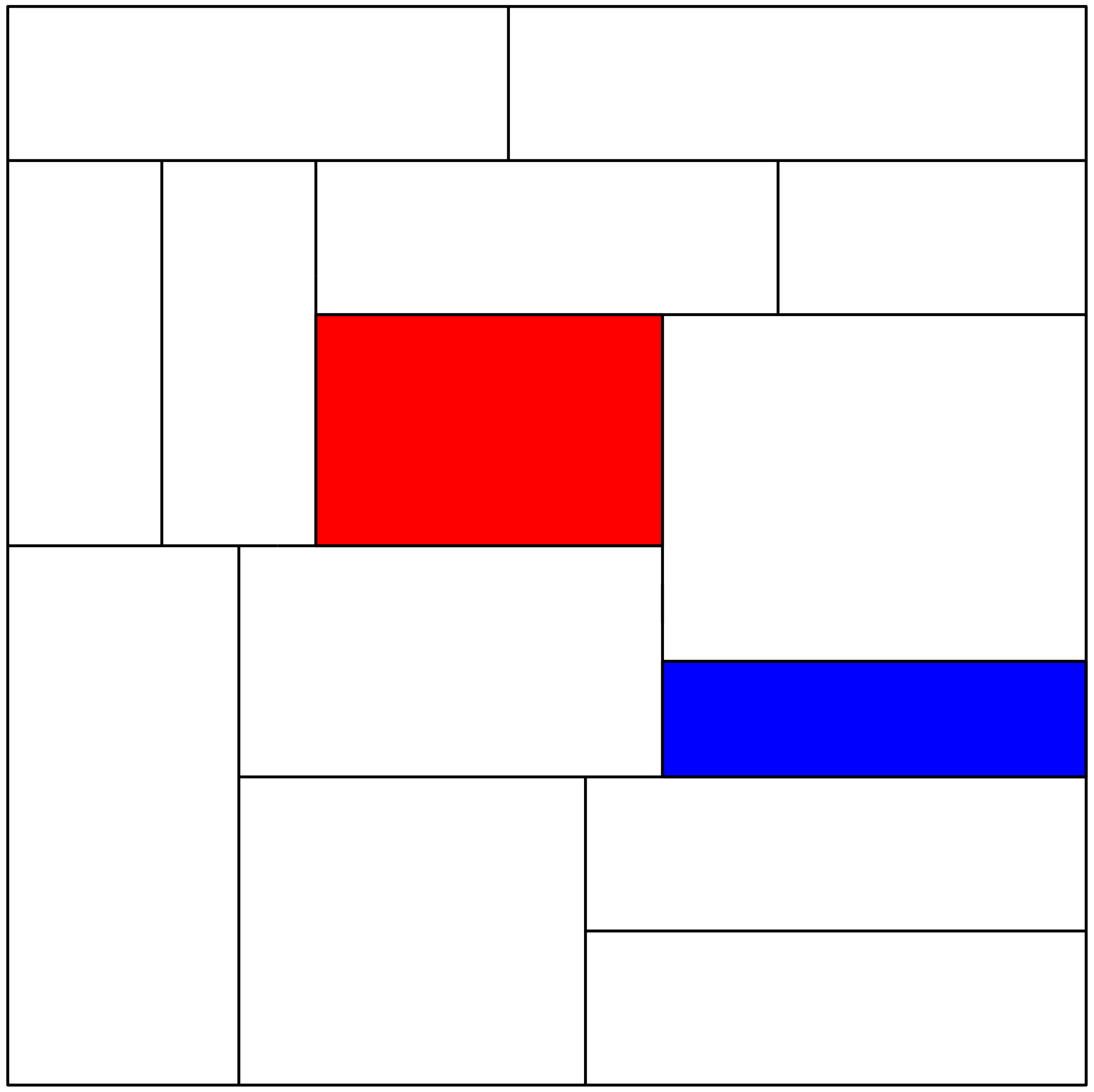
Within $\mathcal{R}$, two rectangles are *left/right neighbors* if they are seperated by a single vertical segment.



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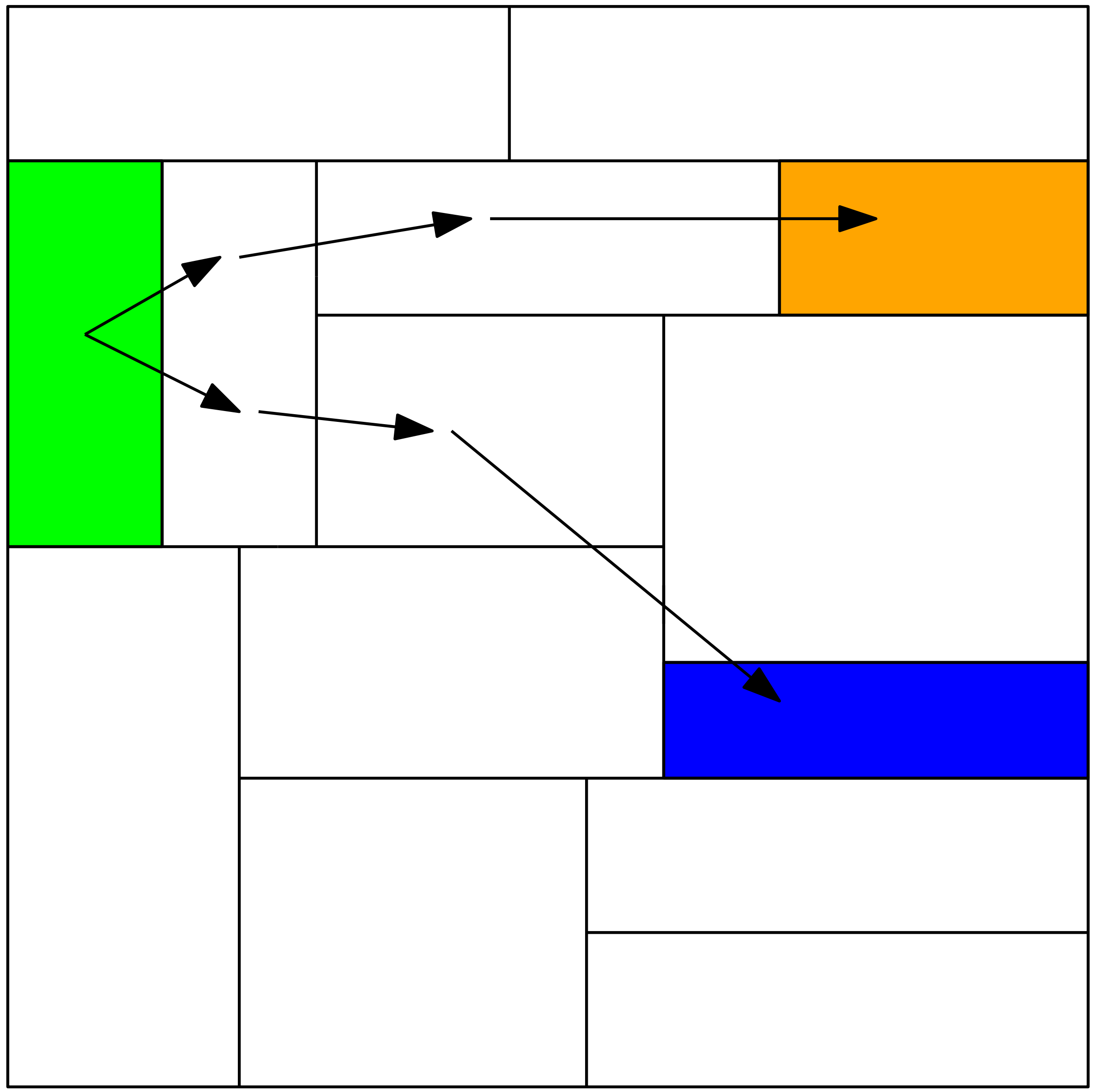
Rectangle r is *left of* rectangle s if they are part of a sequence of left/right neighbors.



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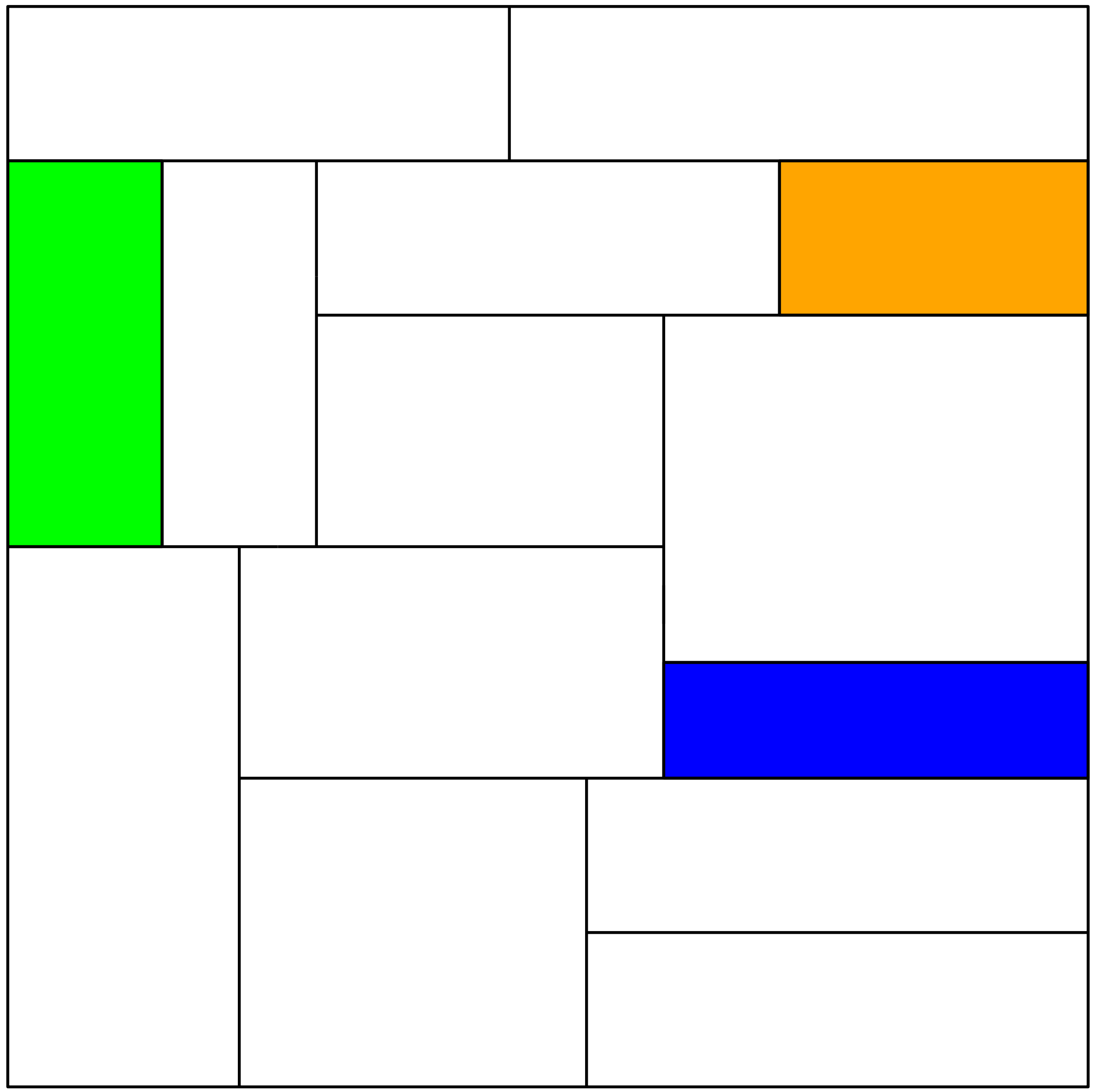

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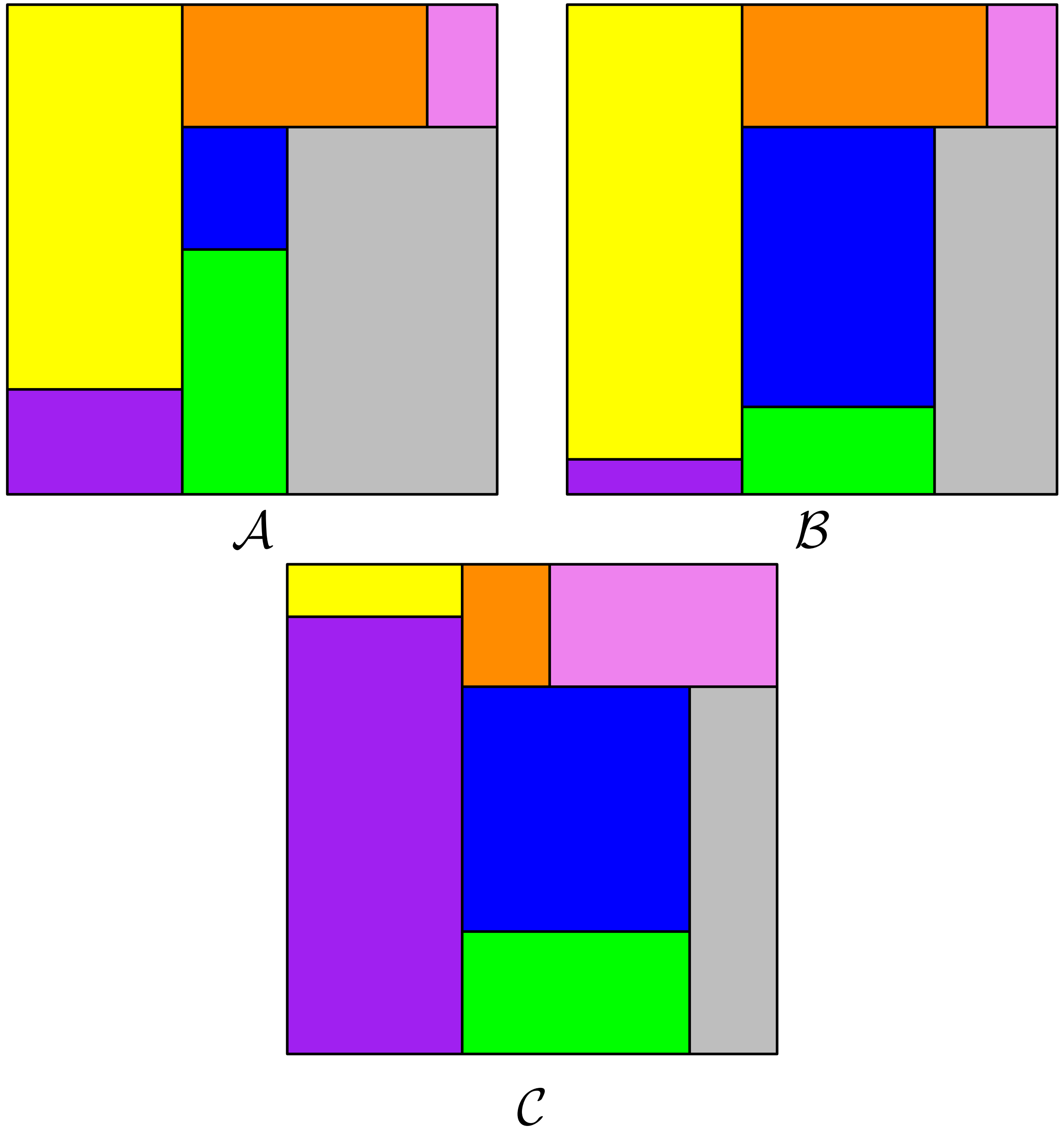
They are *strongly equivalent* if they also preserve contact between rectangles.



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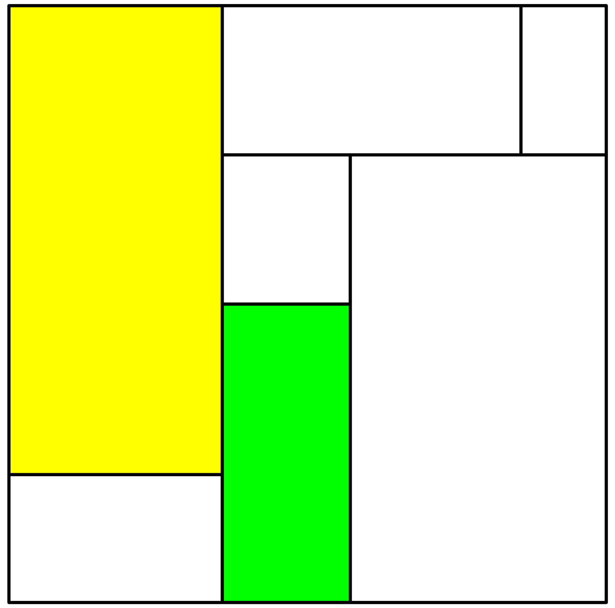
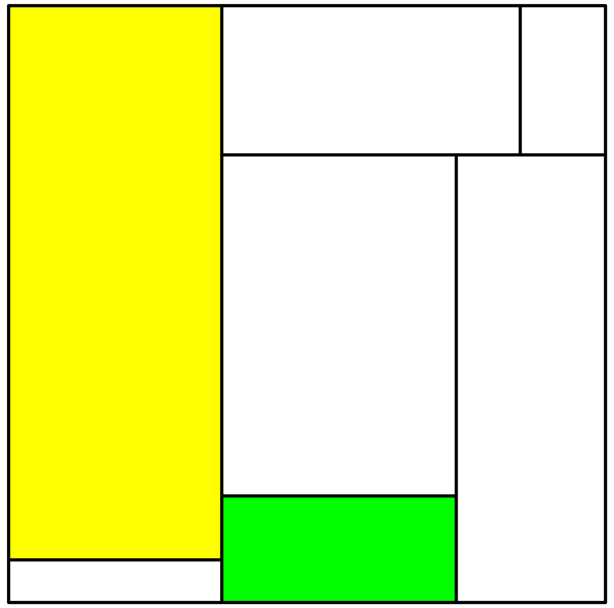
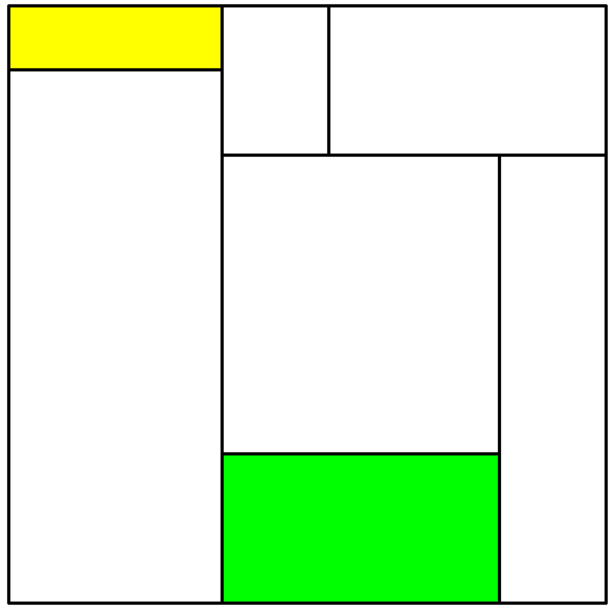


$\mathcal{A}$



$\mathcal{B}$



$\mathcal{C}$

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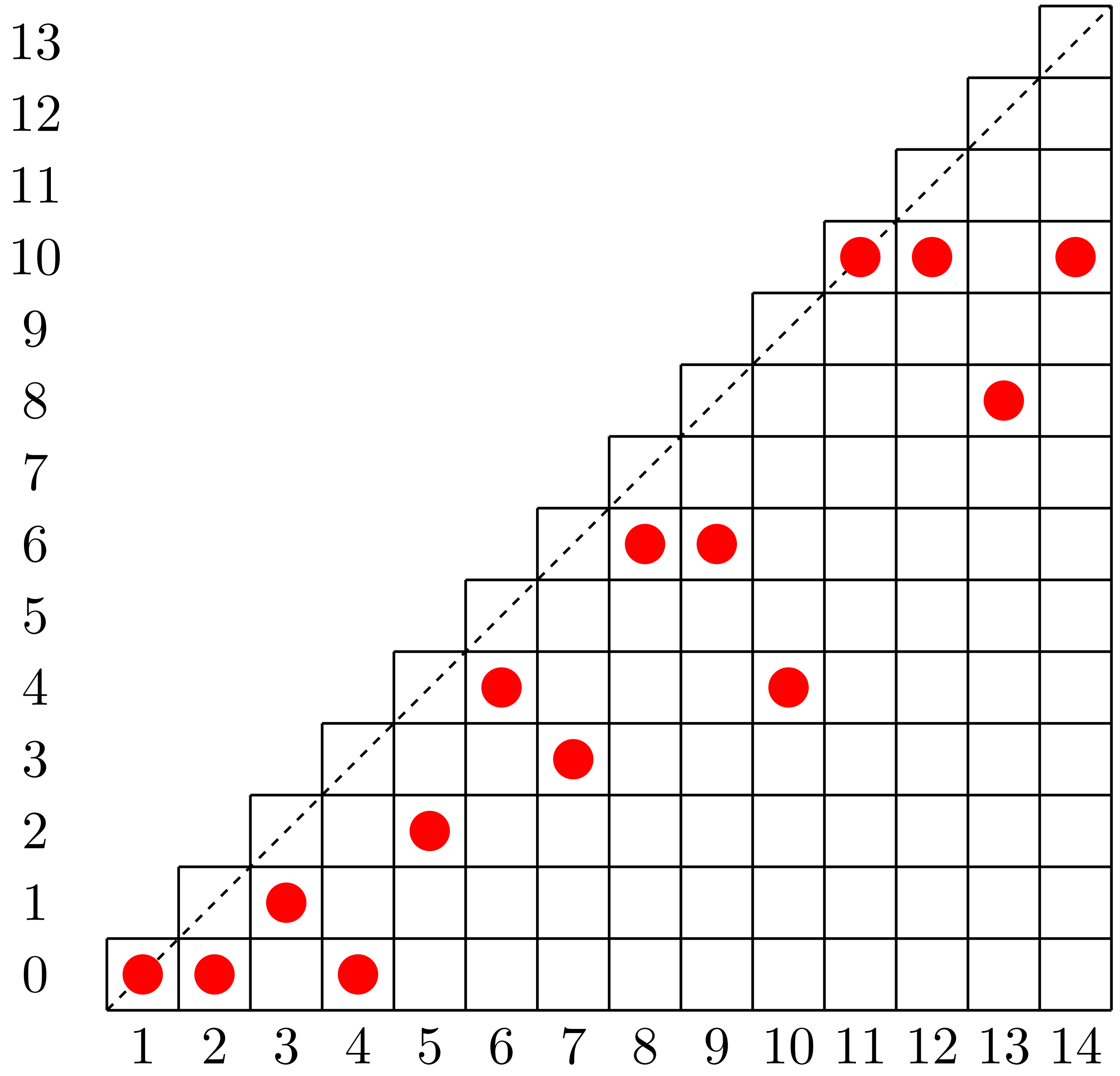

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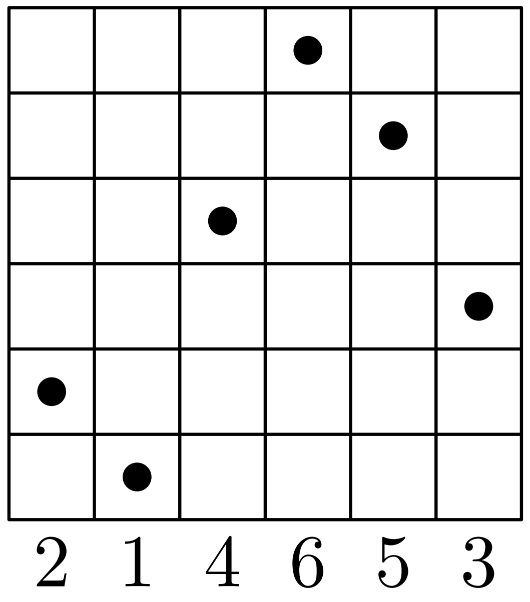

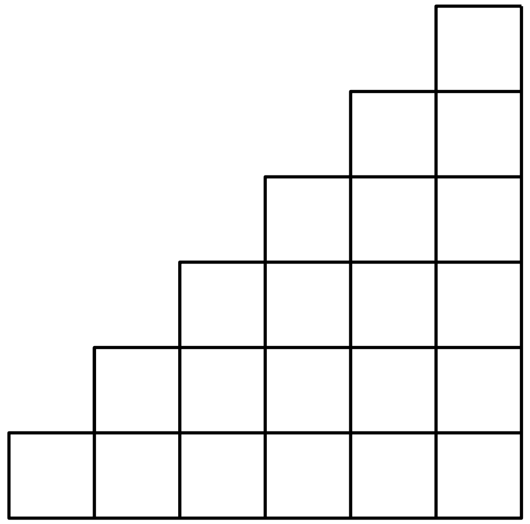

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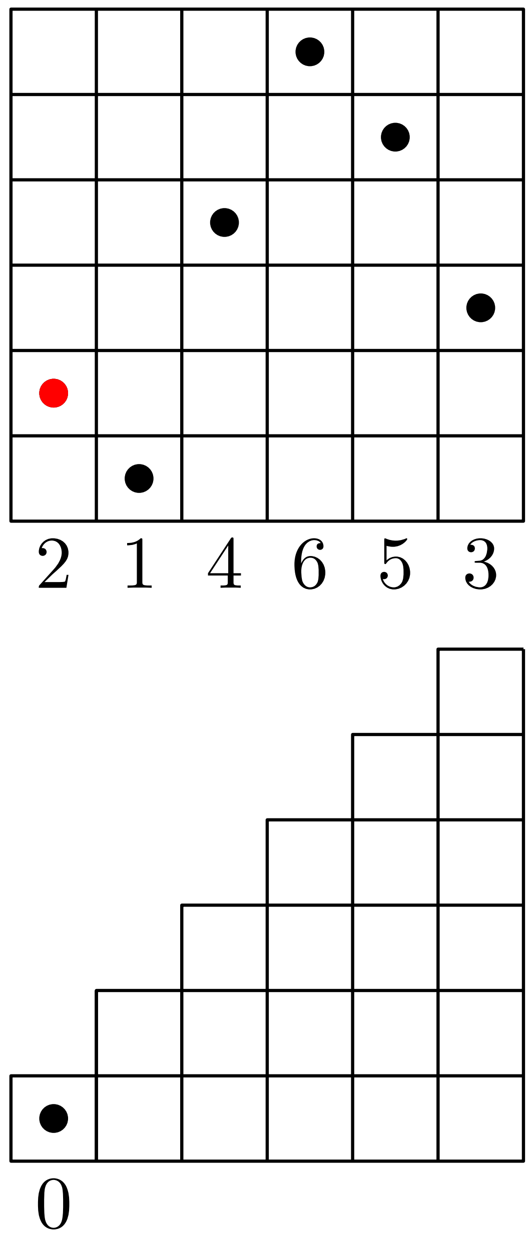

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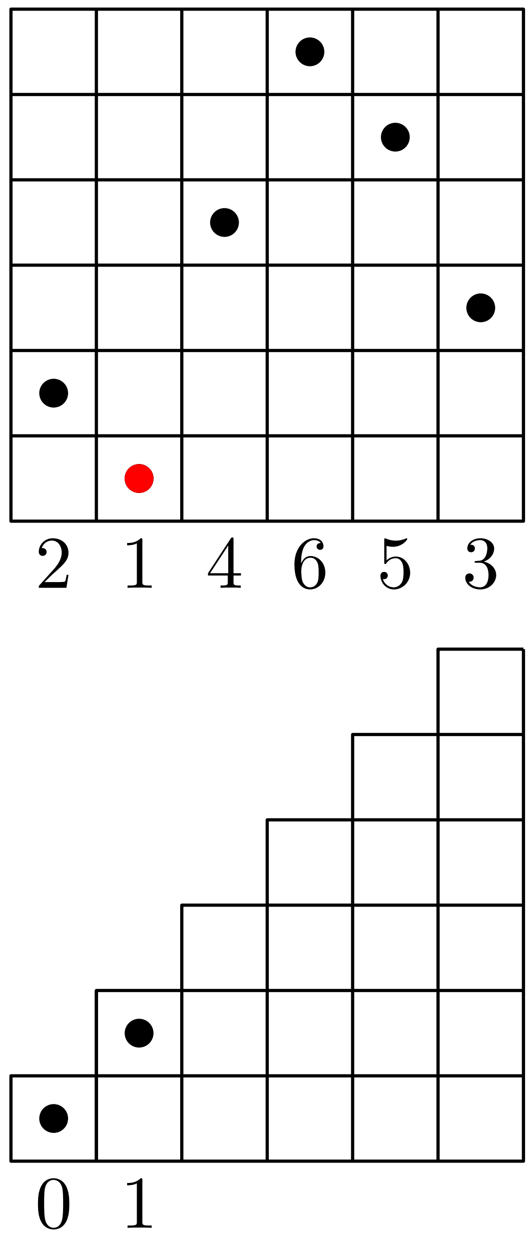

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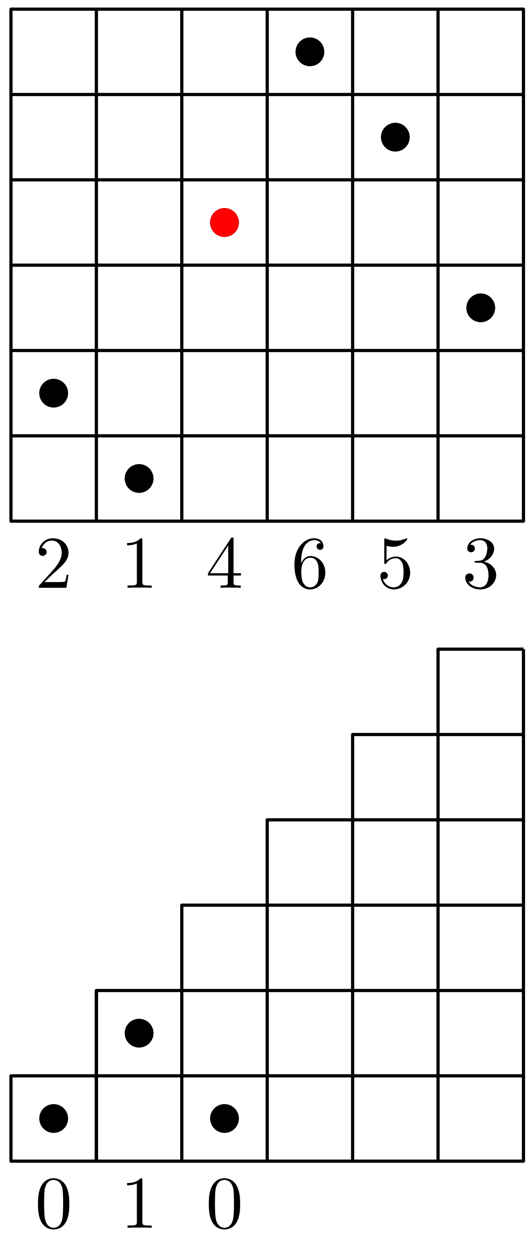

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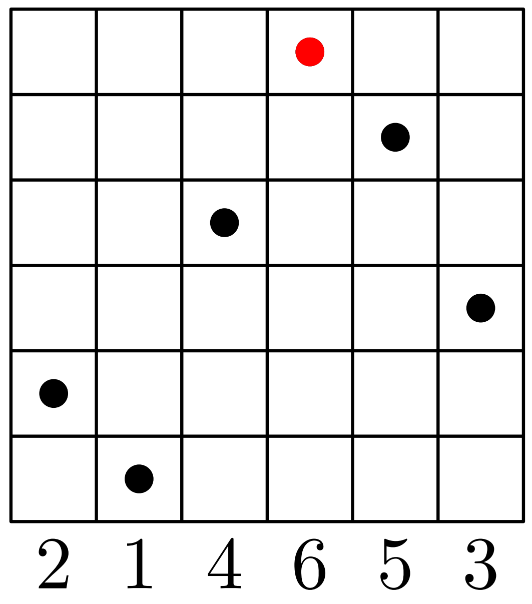

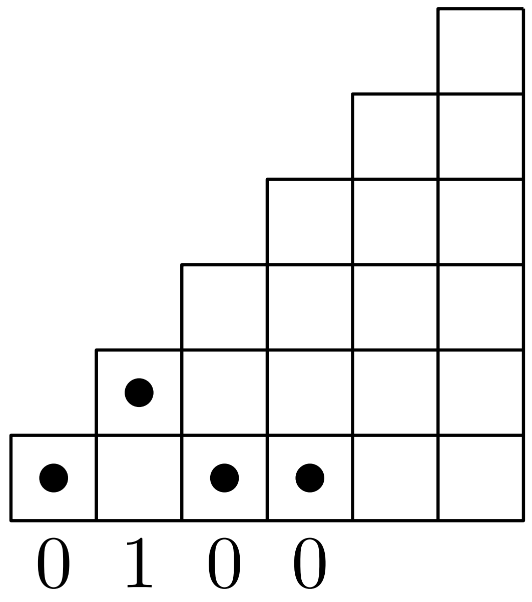

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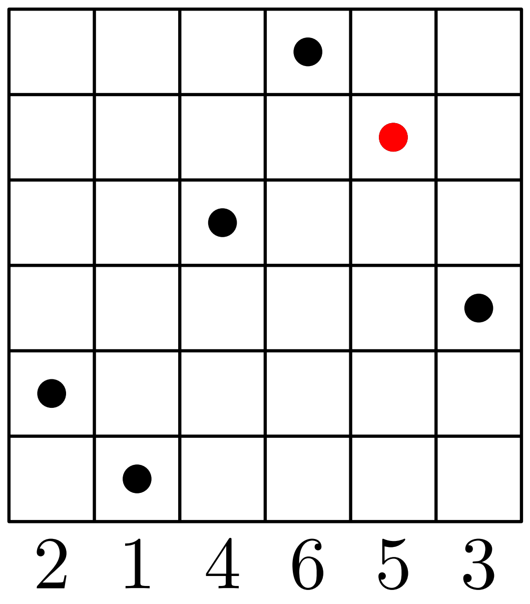

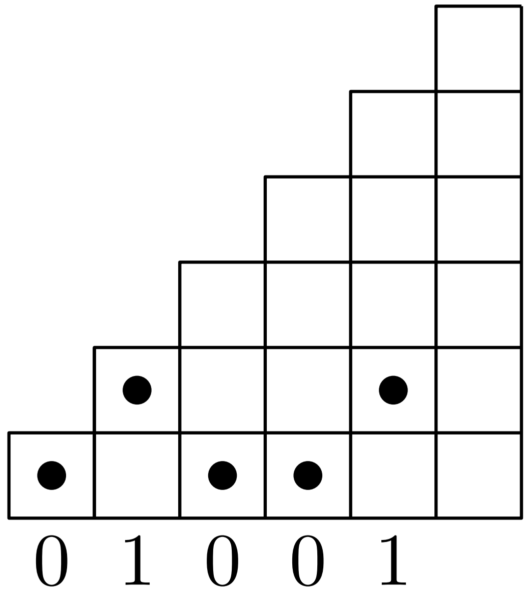

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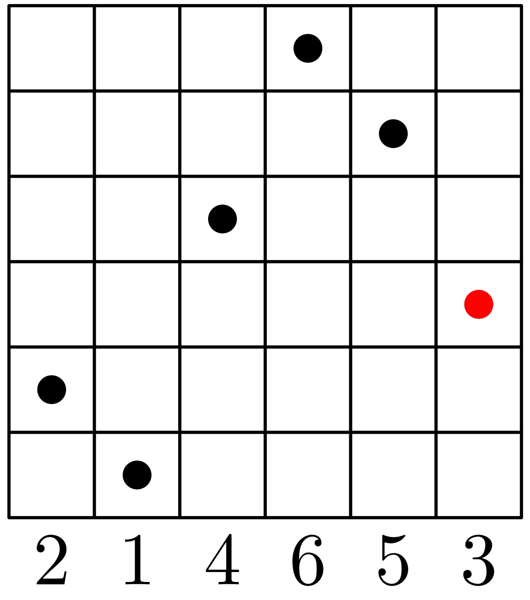

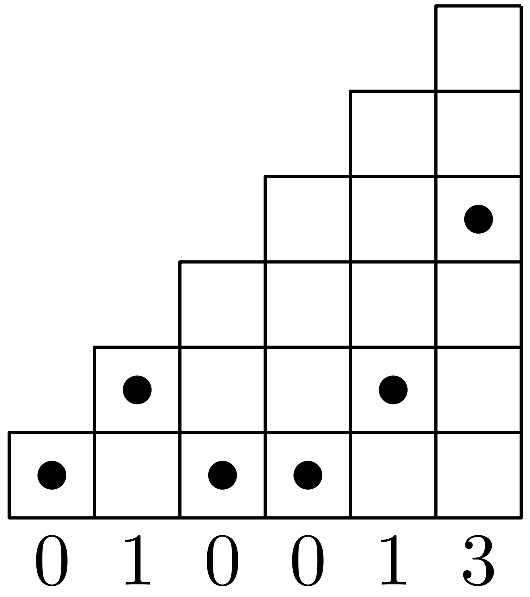

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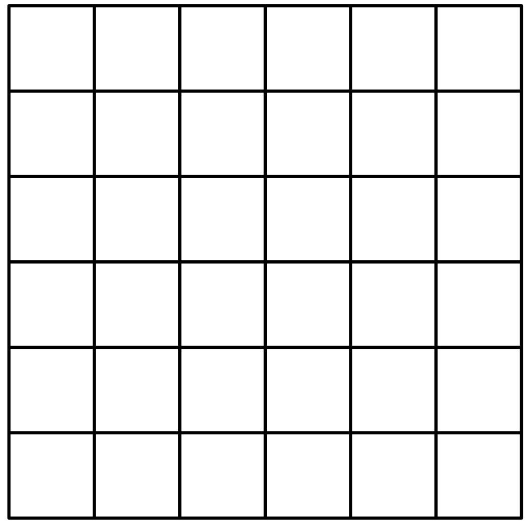

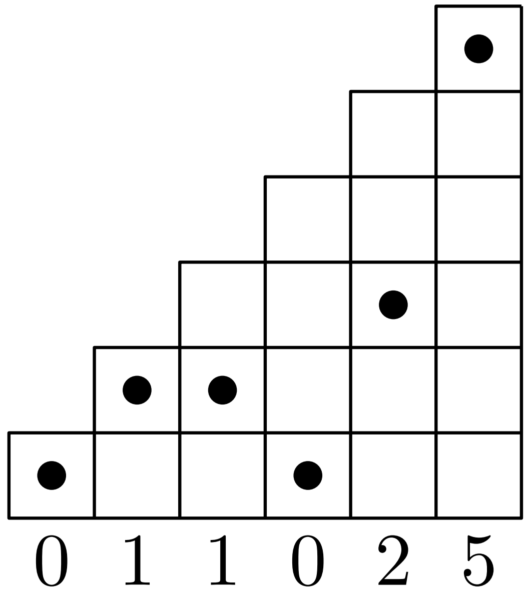

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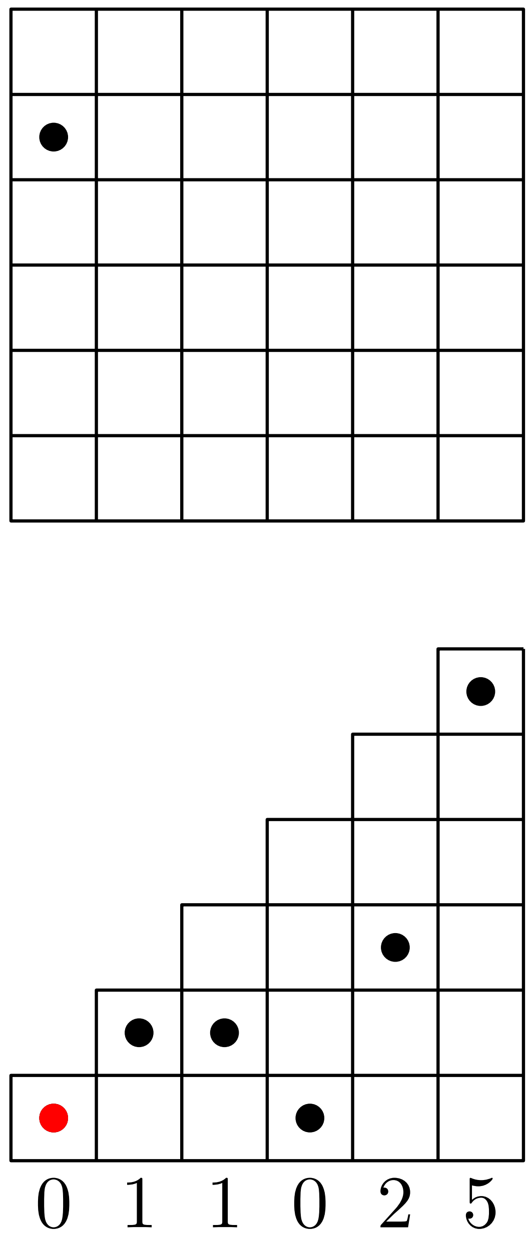

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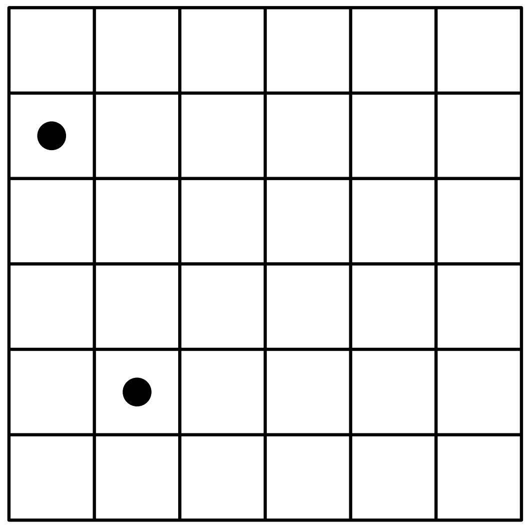

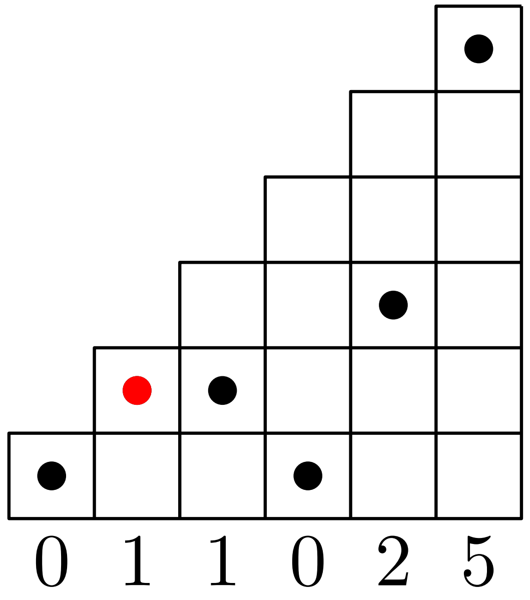

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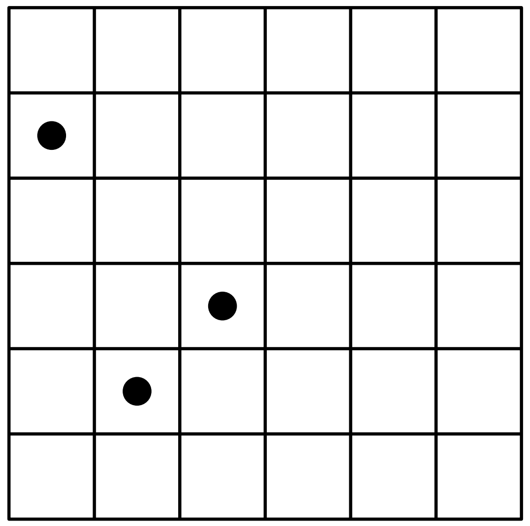

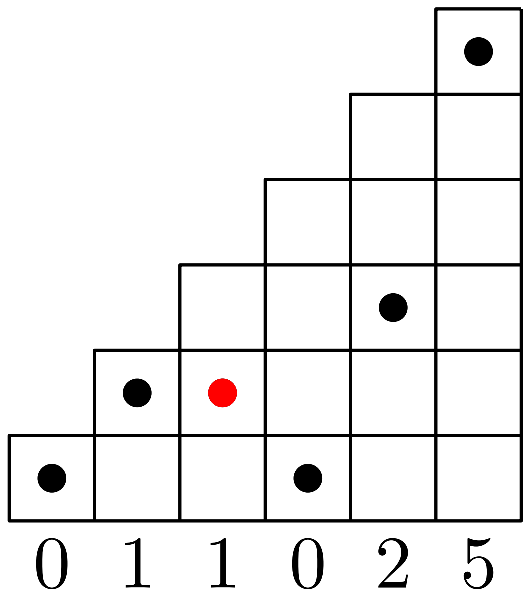

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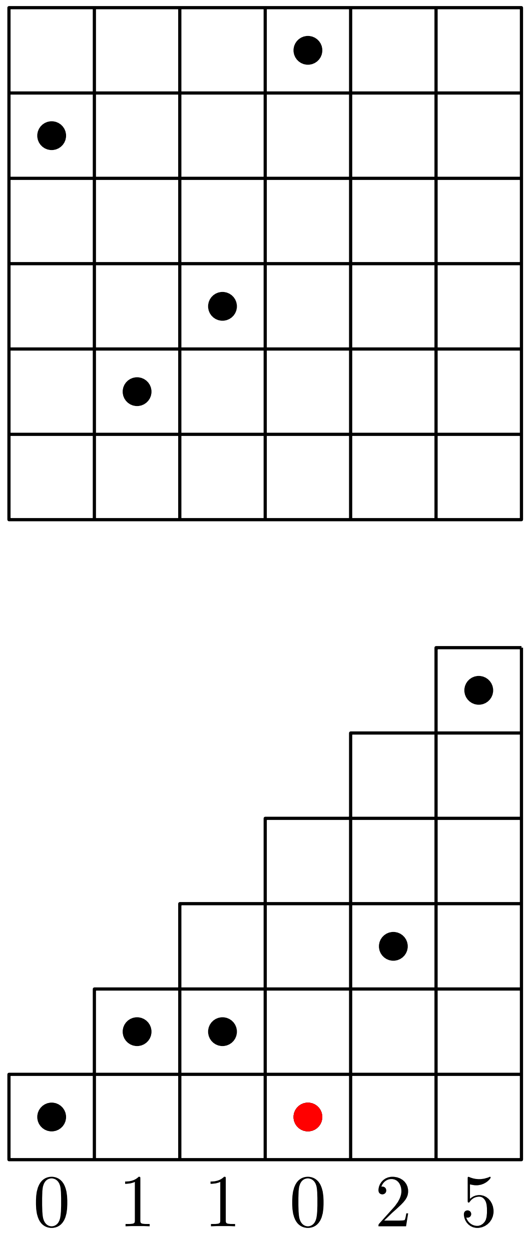

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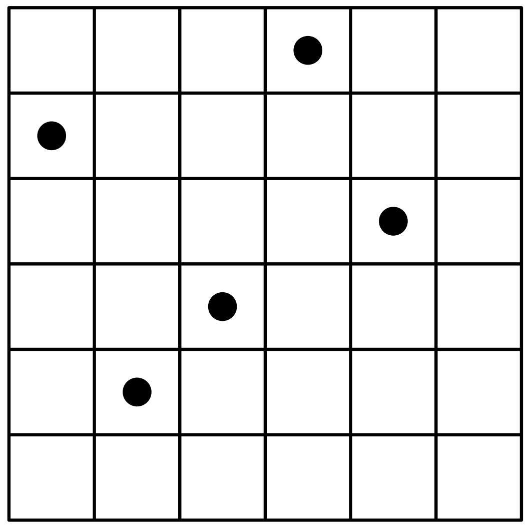

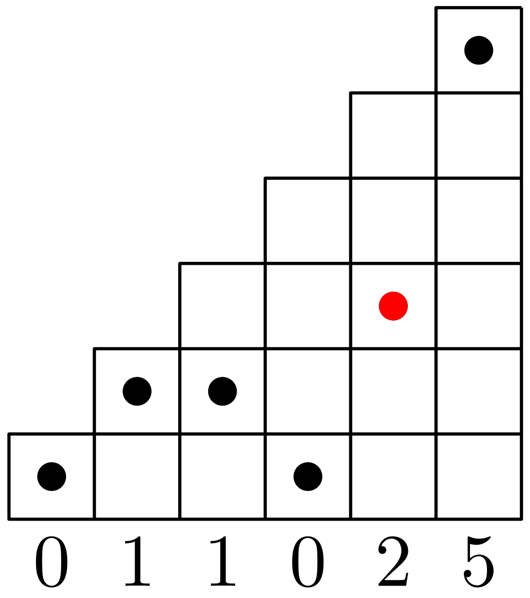

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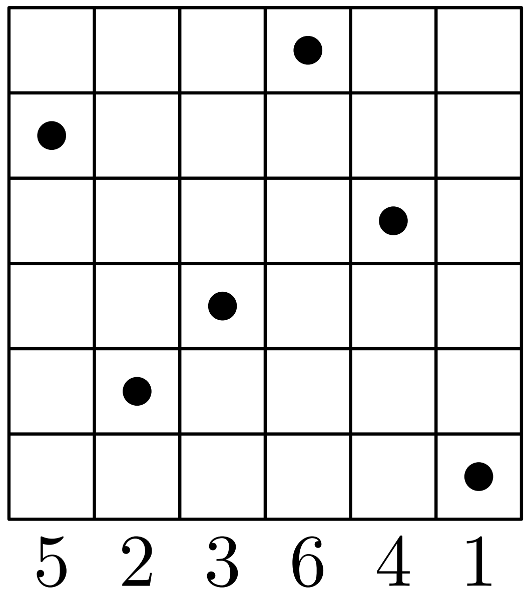

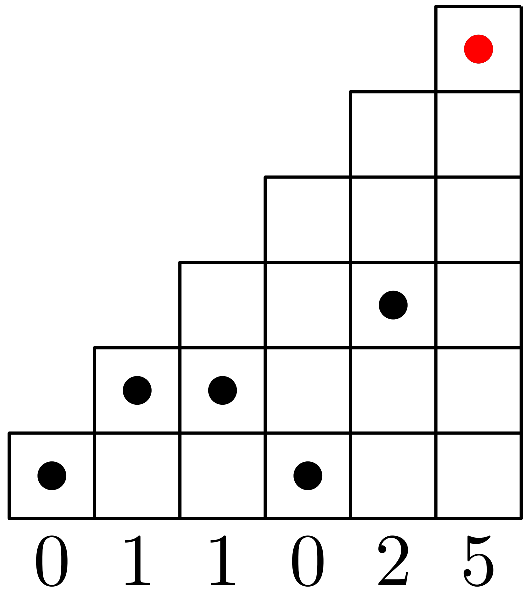

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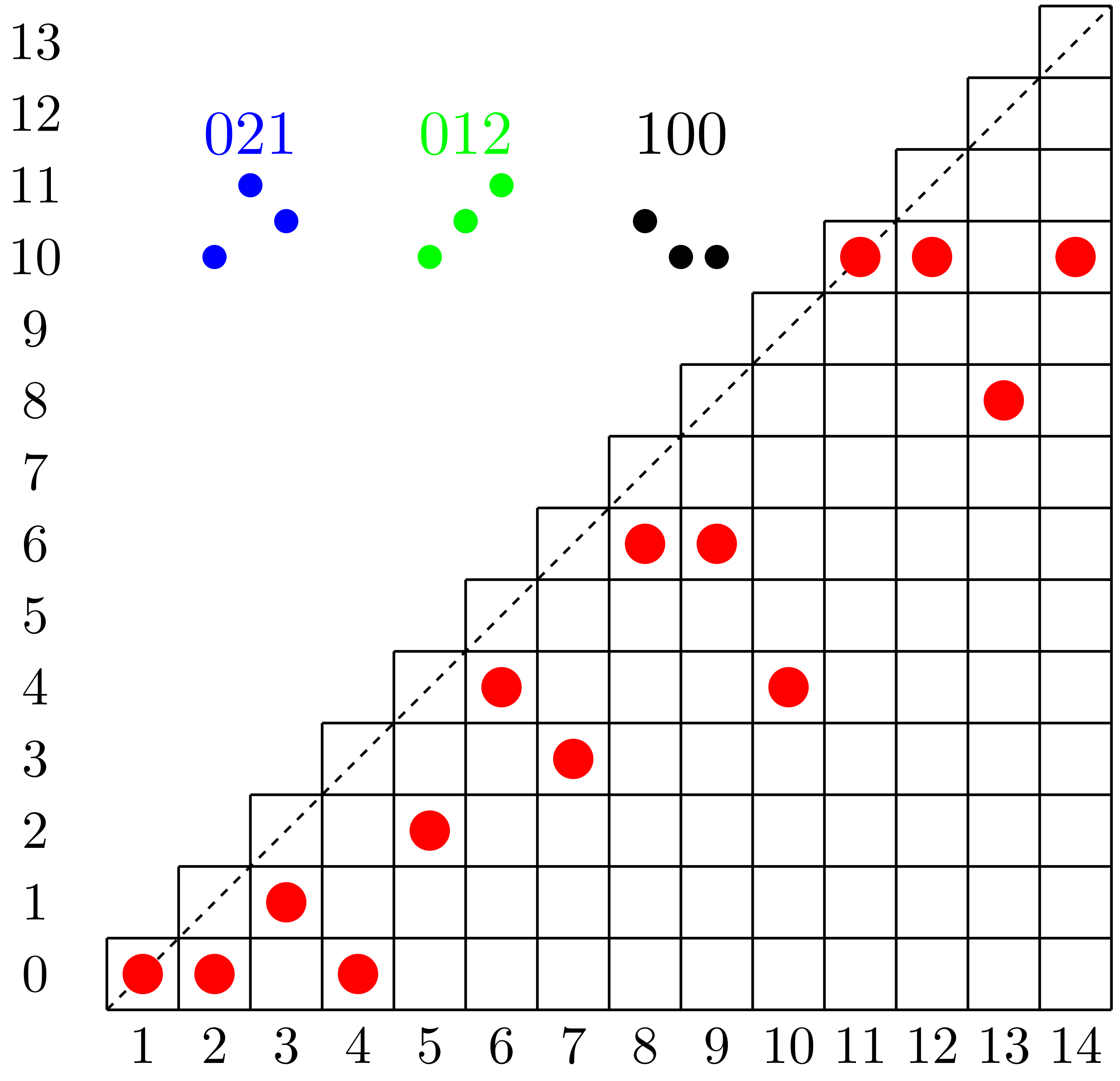

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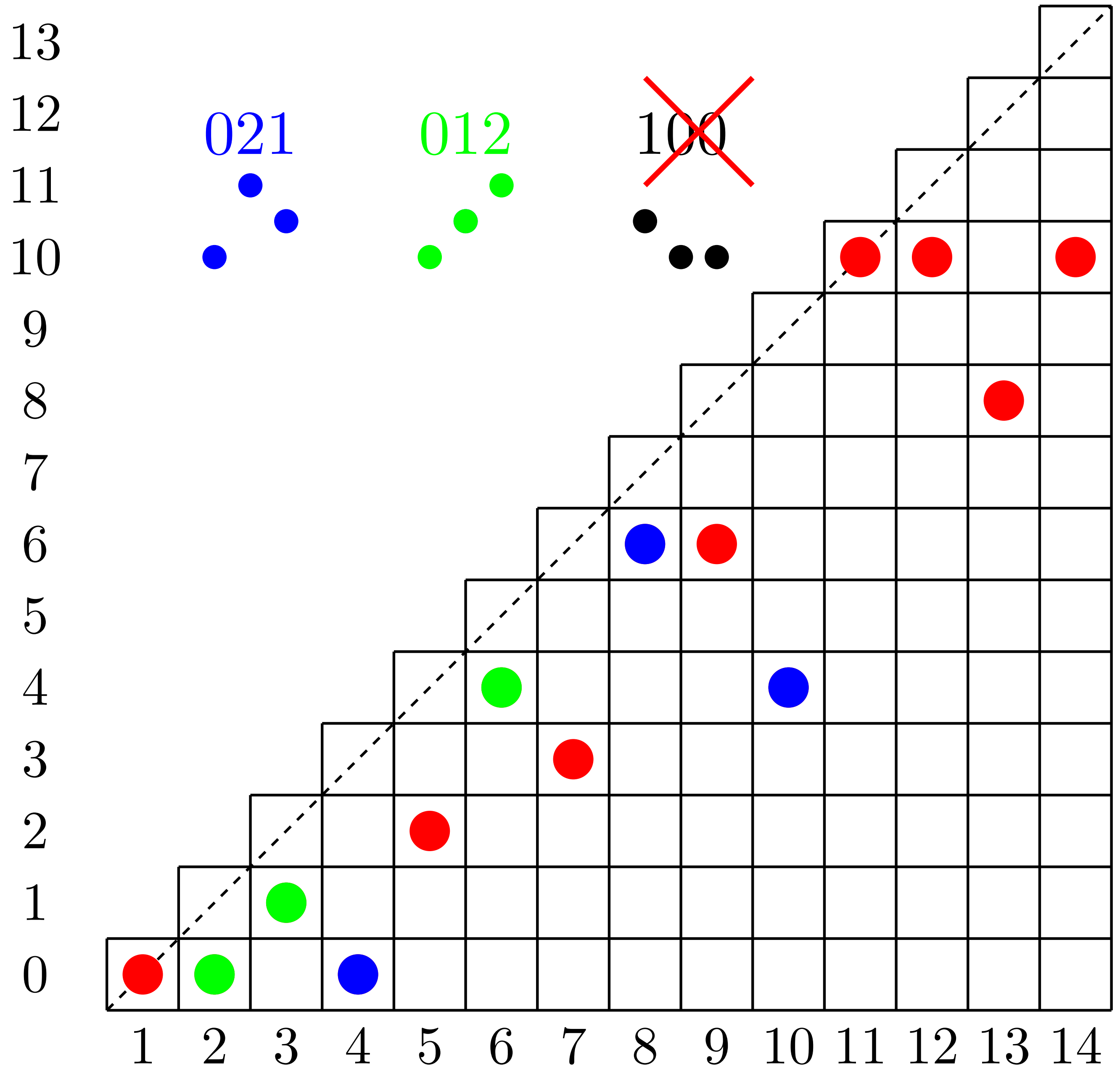

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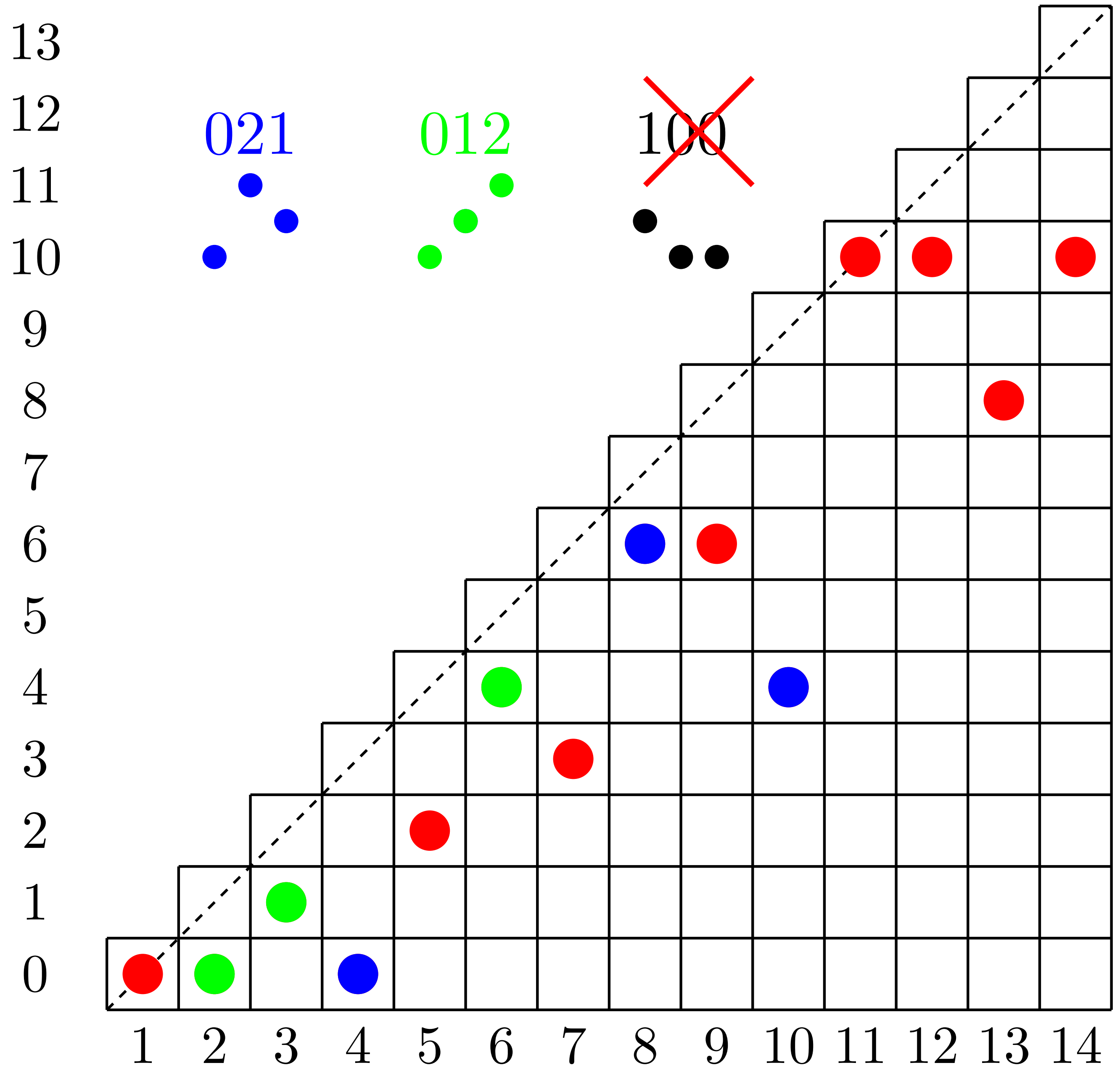
If s does not contain t, then we say that s avoids t. Denote by $I_n(L)$ the set of inversion sequences of length n which avoid all of the patterns in L.



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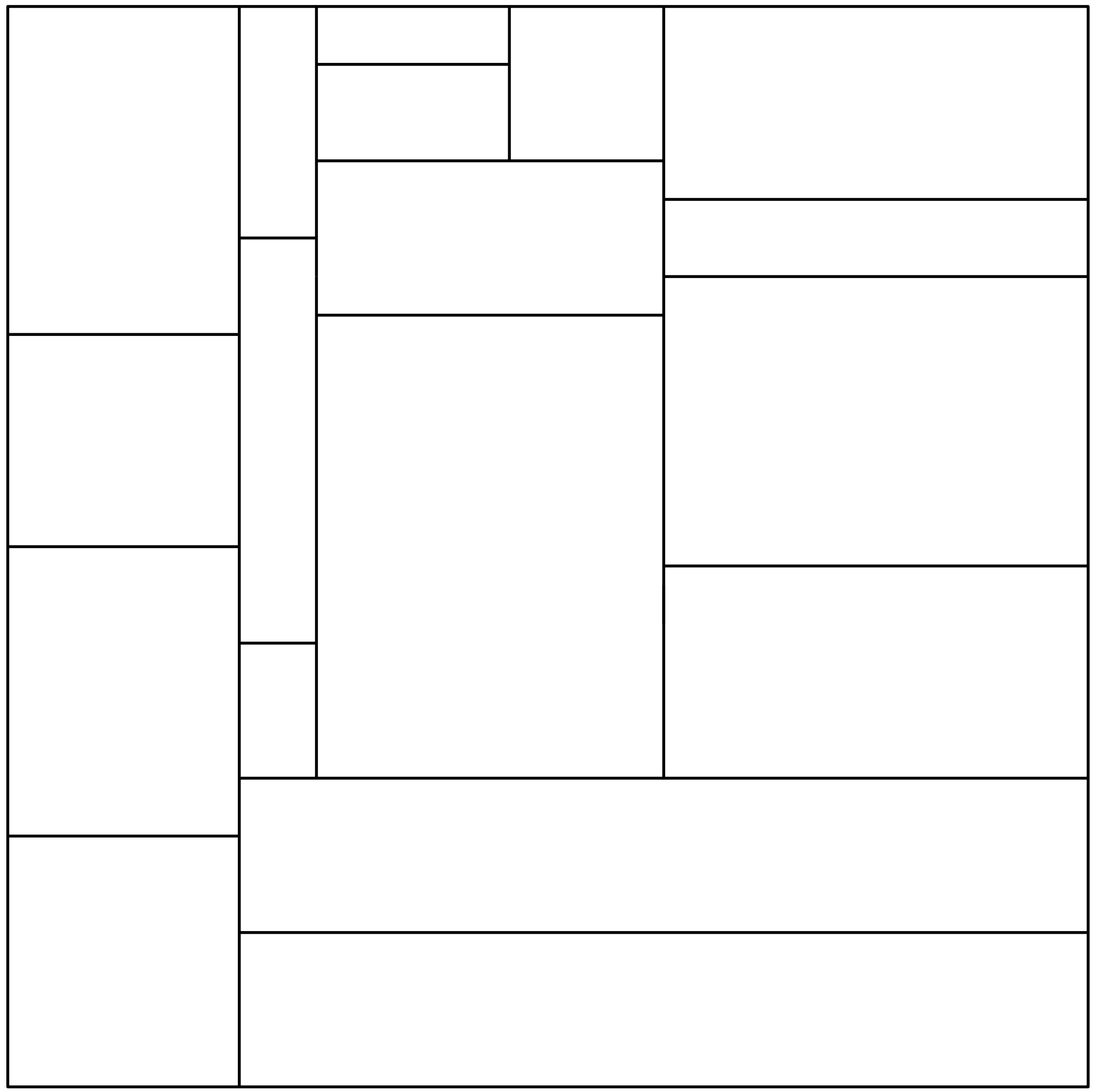

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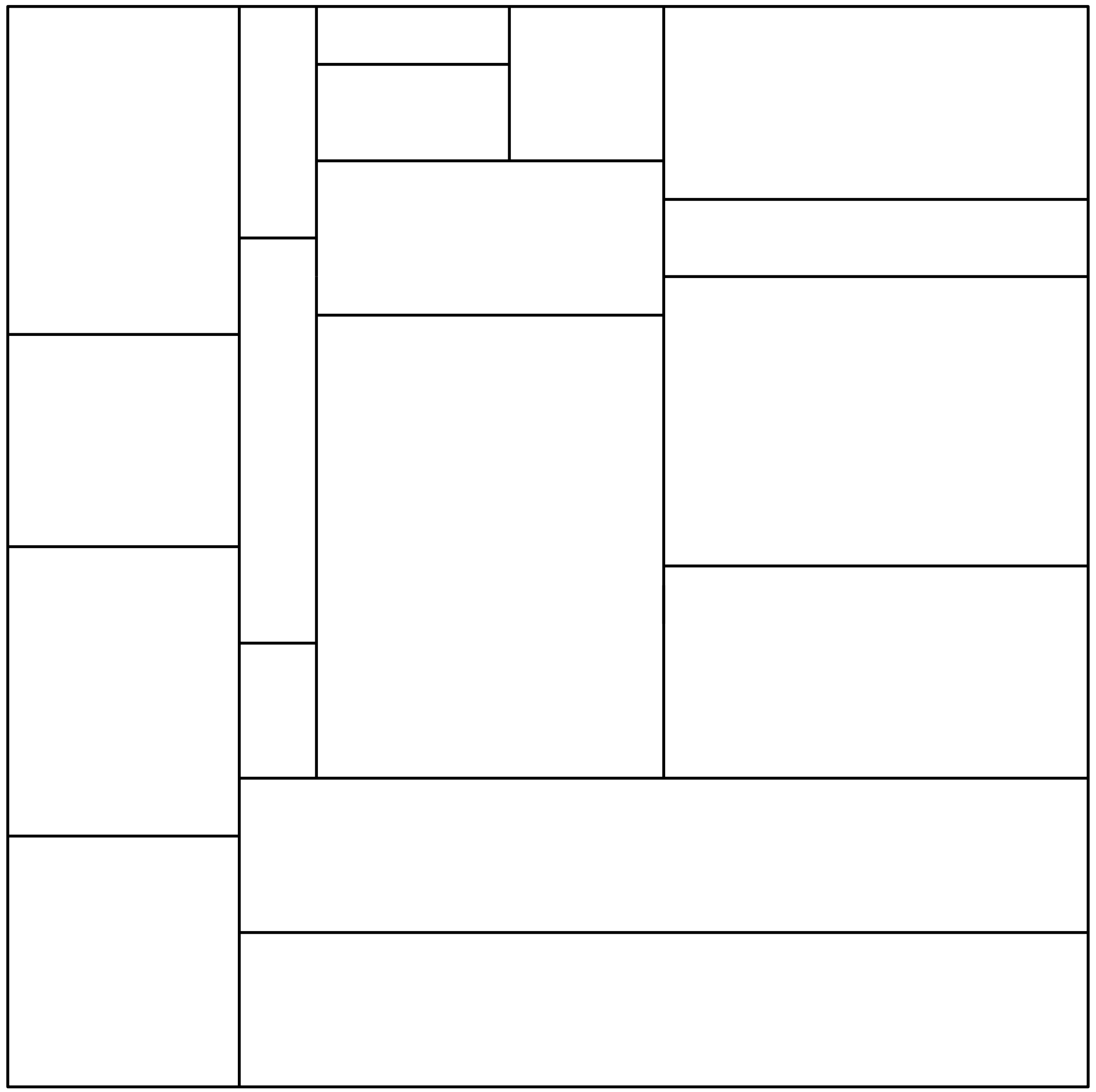

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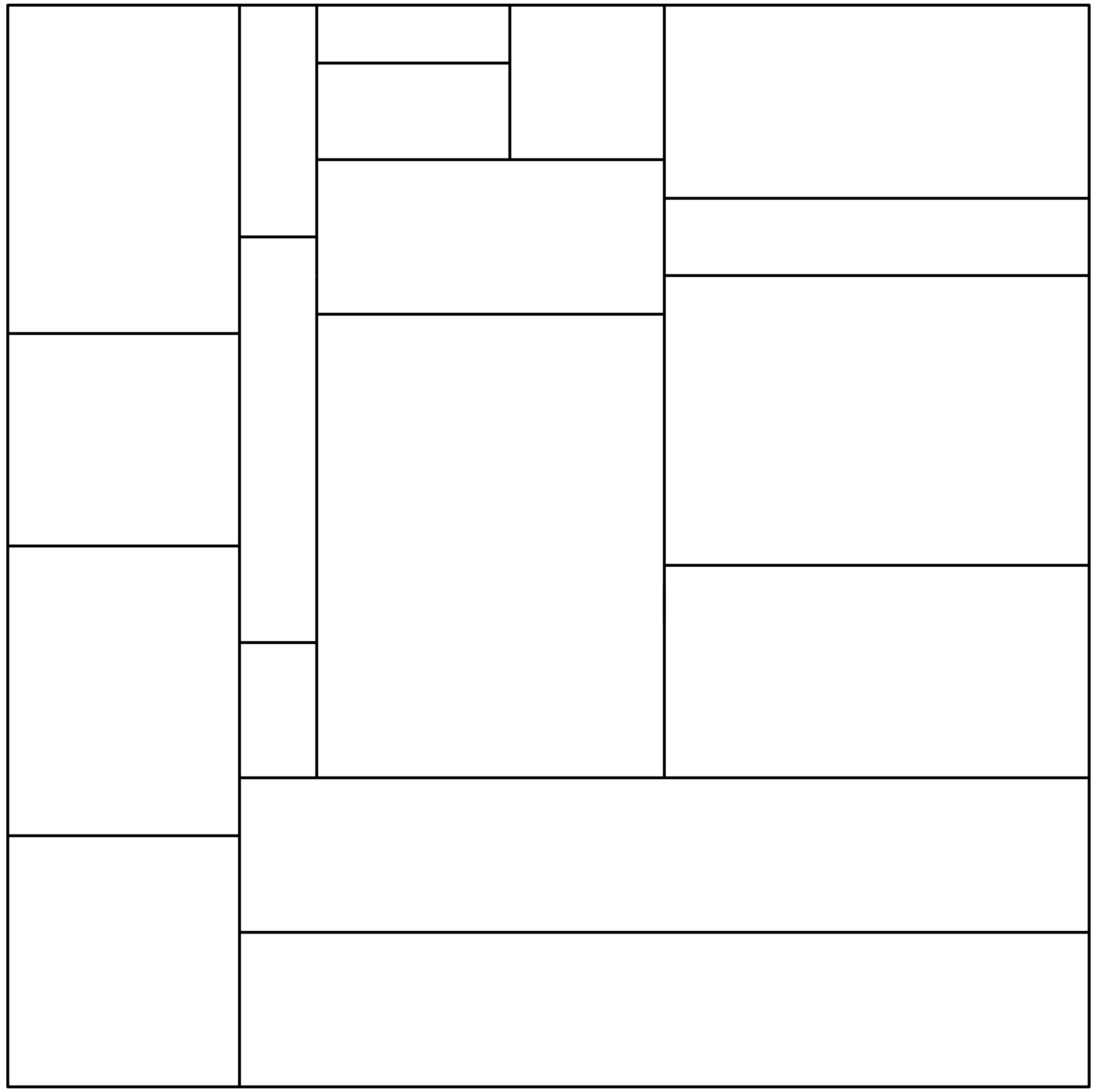
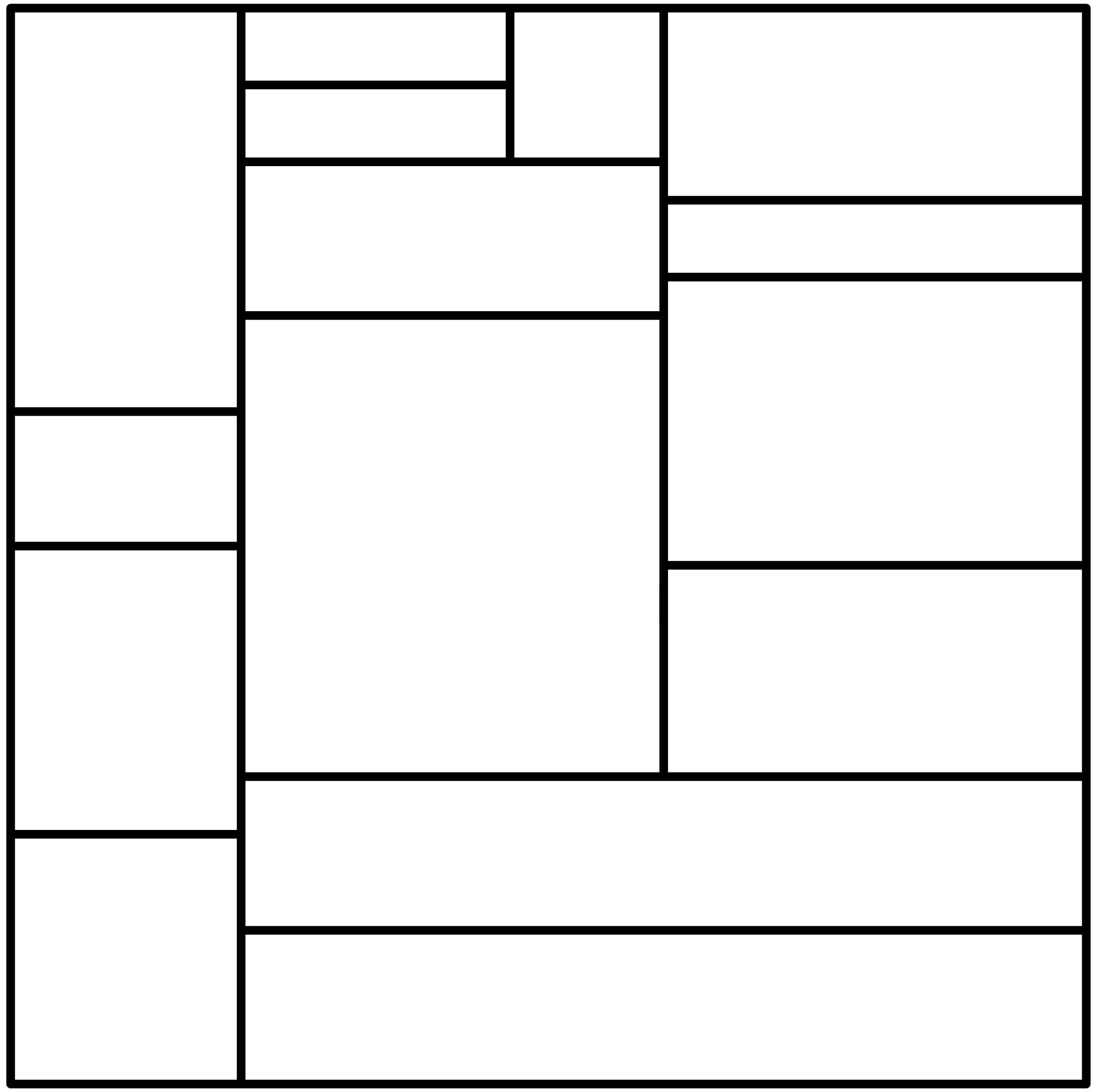
Let L be a set of rectangulation patterns and denote by $R_n^w(L)$ and $R_n^s(L)$ the set of weak and, respectively, strong rectangulations of size n that avoid all patterns in L.

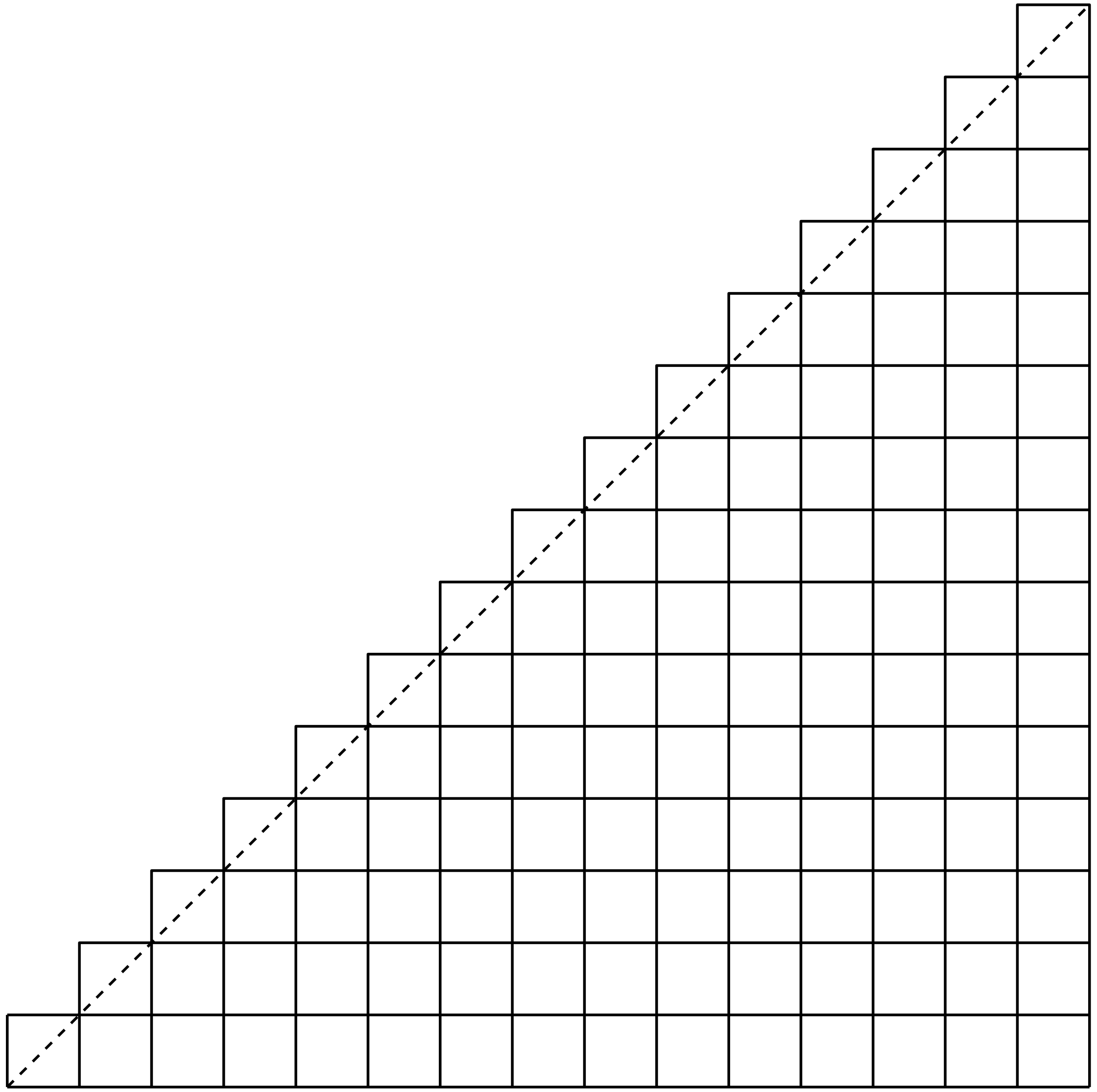
Our results cover all the (essentially different) cases where $L \subseteq \{\top, \perp, \vdash, \dashv\}$.



$|R_n^w(\top)| = C_n$ (Williams)

Proof: Bijection to Dyck paths via non-decreasing inversion sequences





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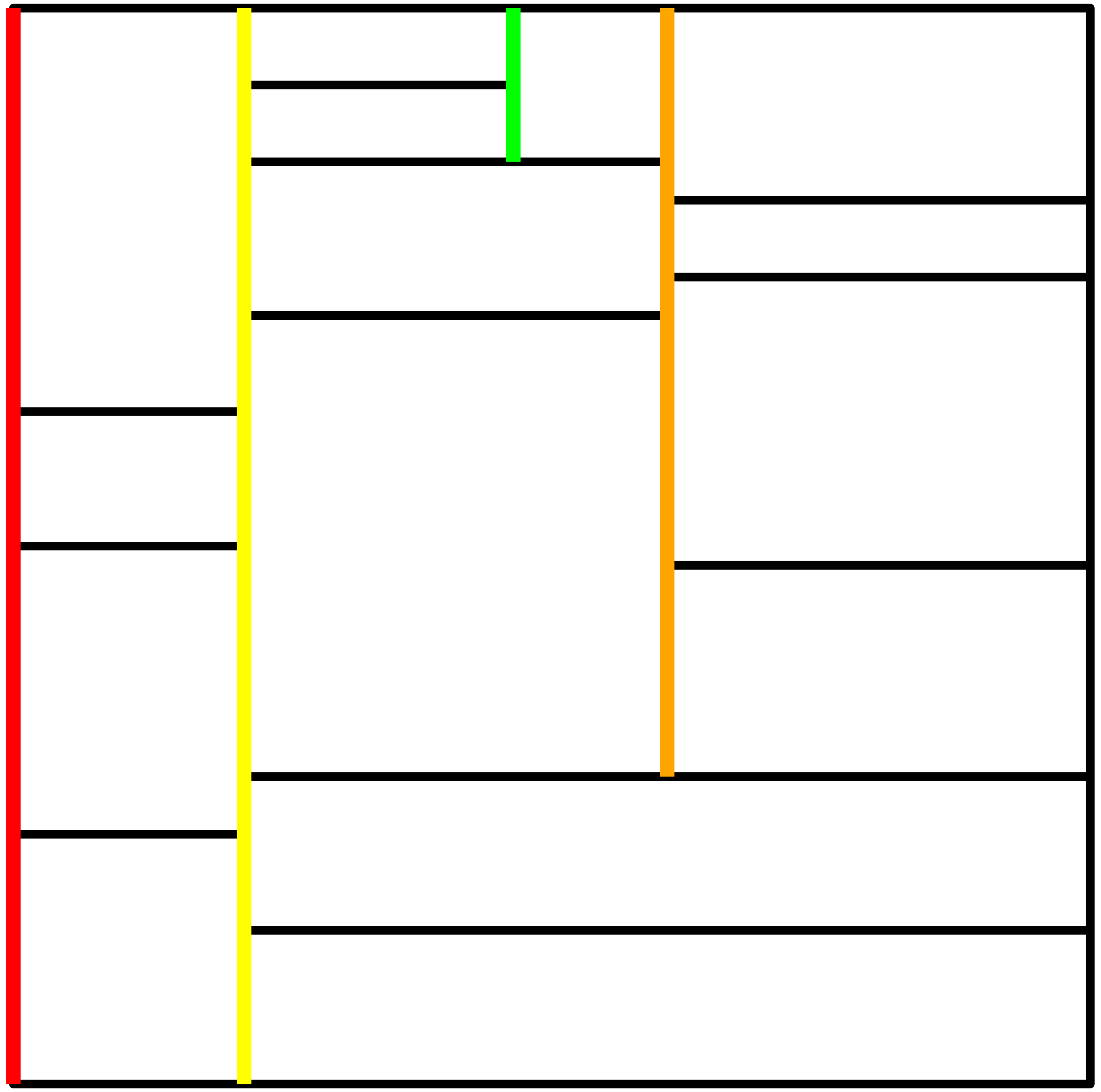

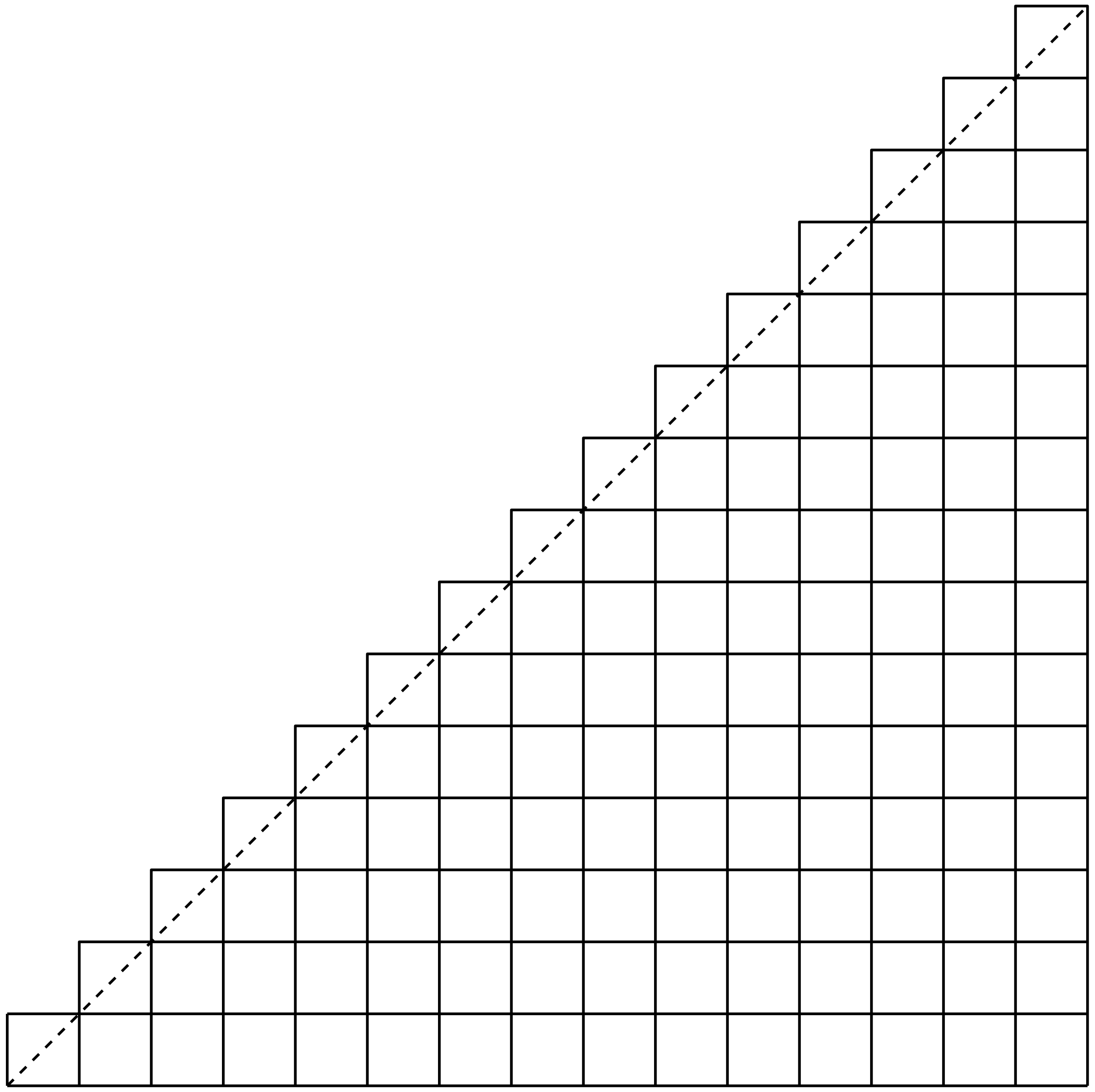

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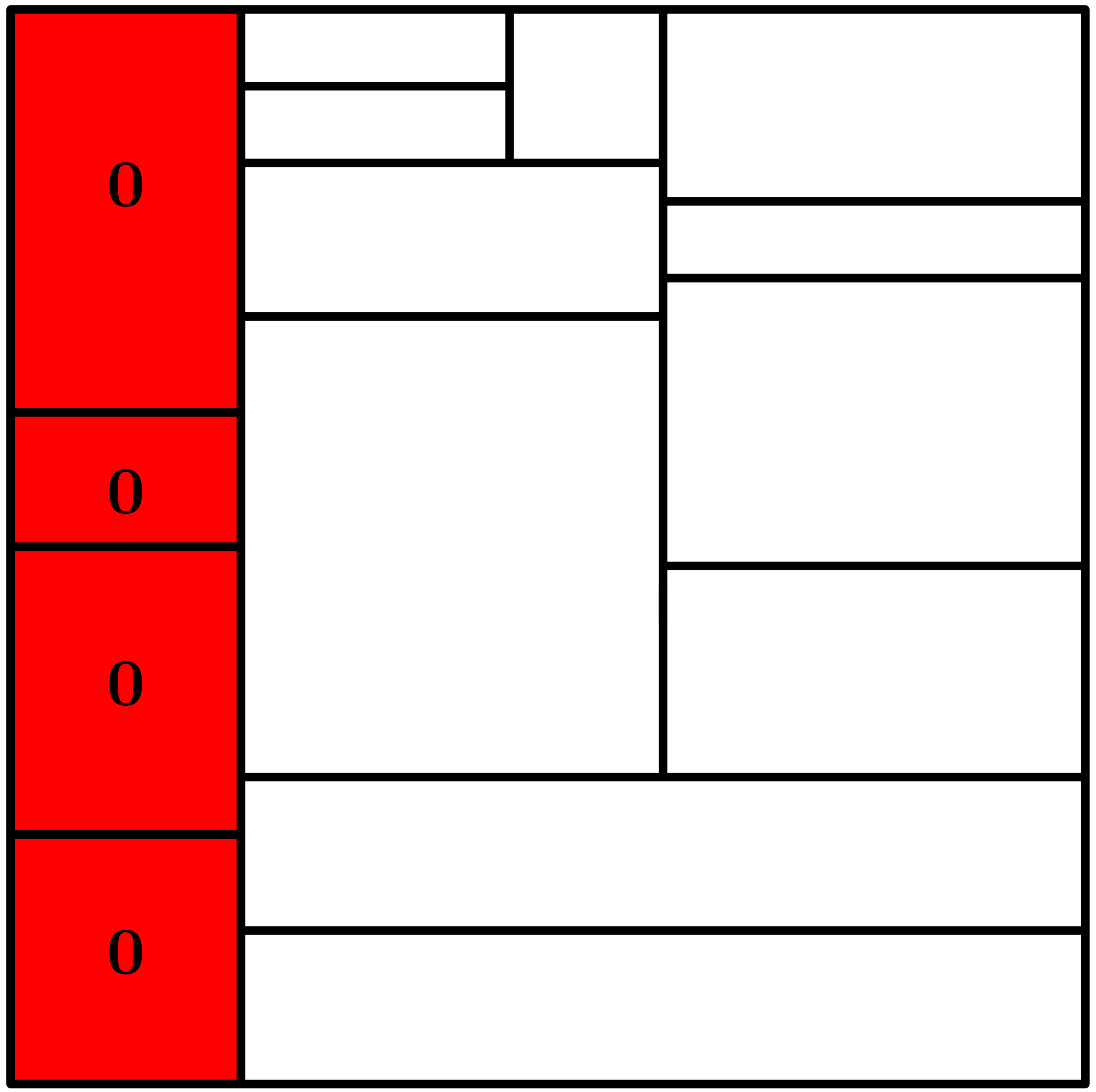

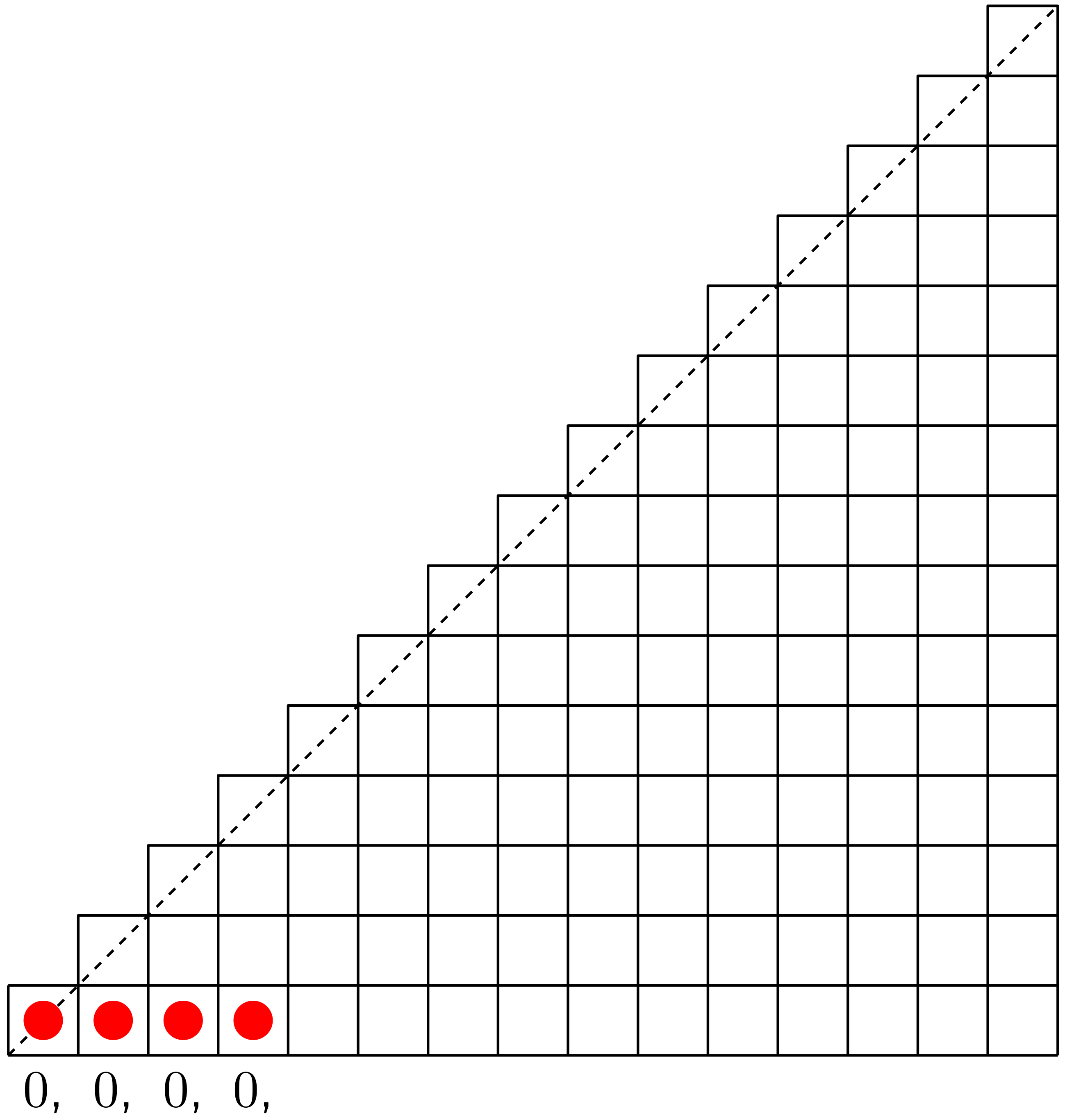

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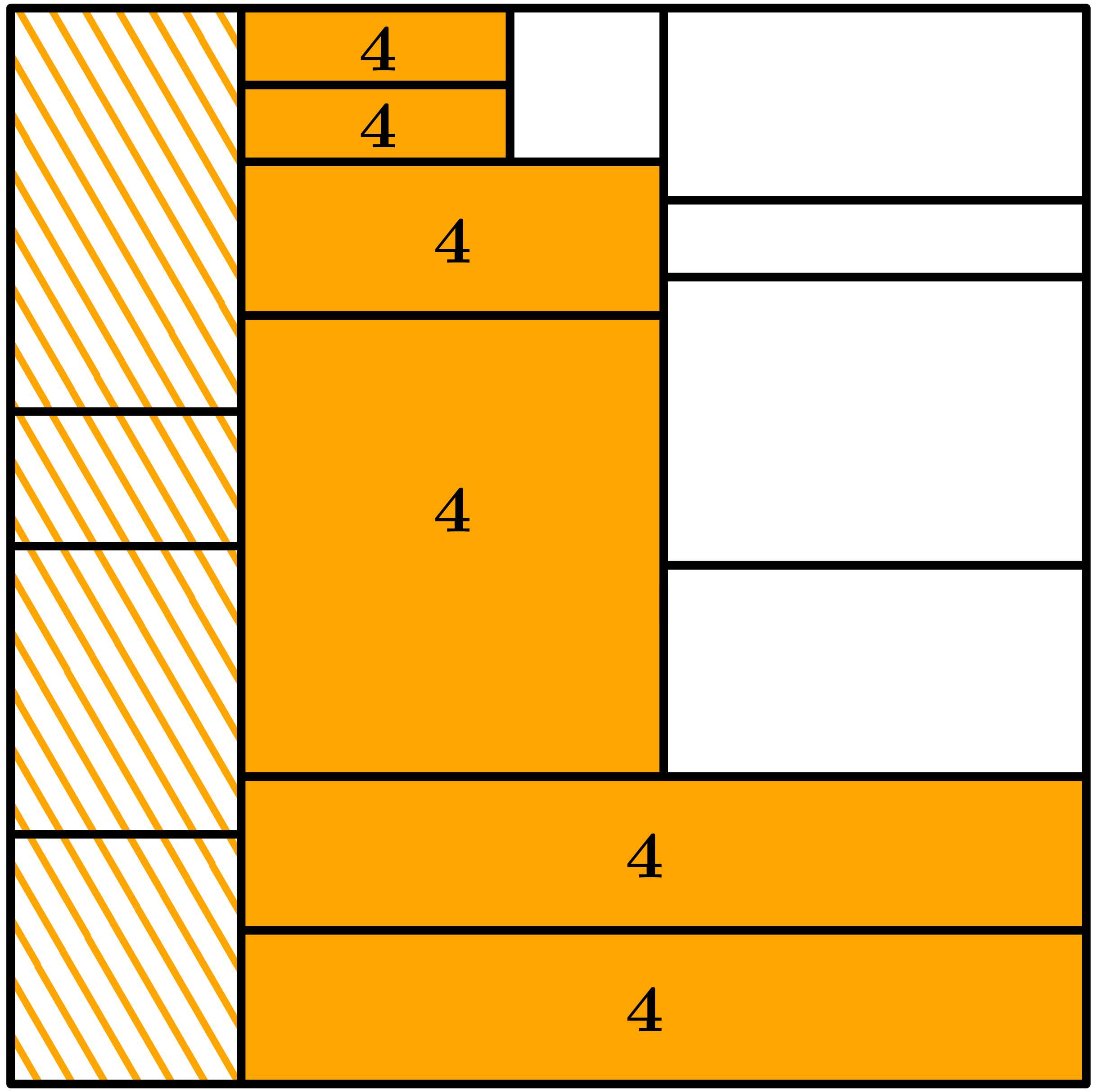

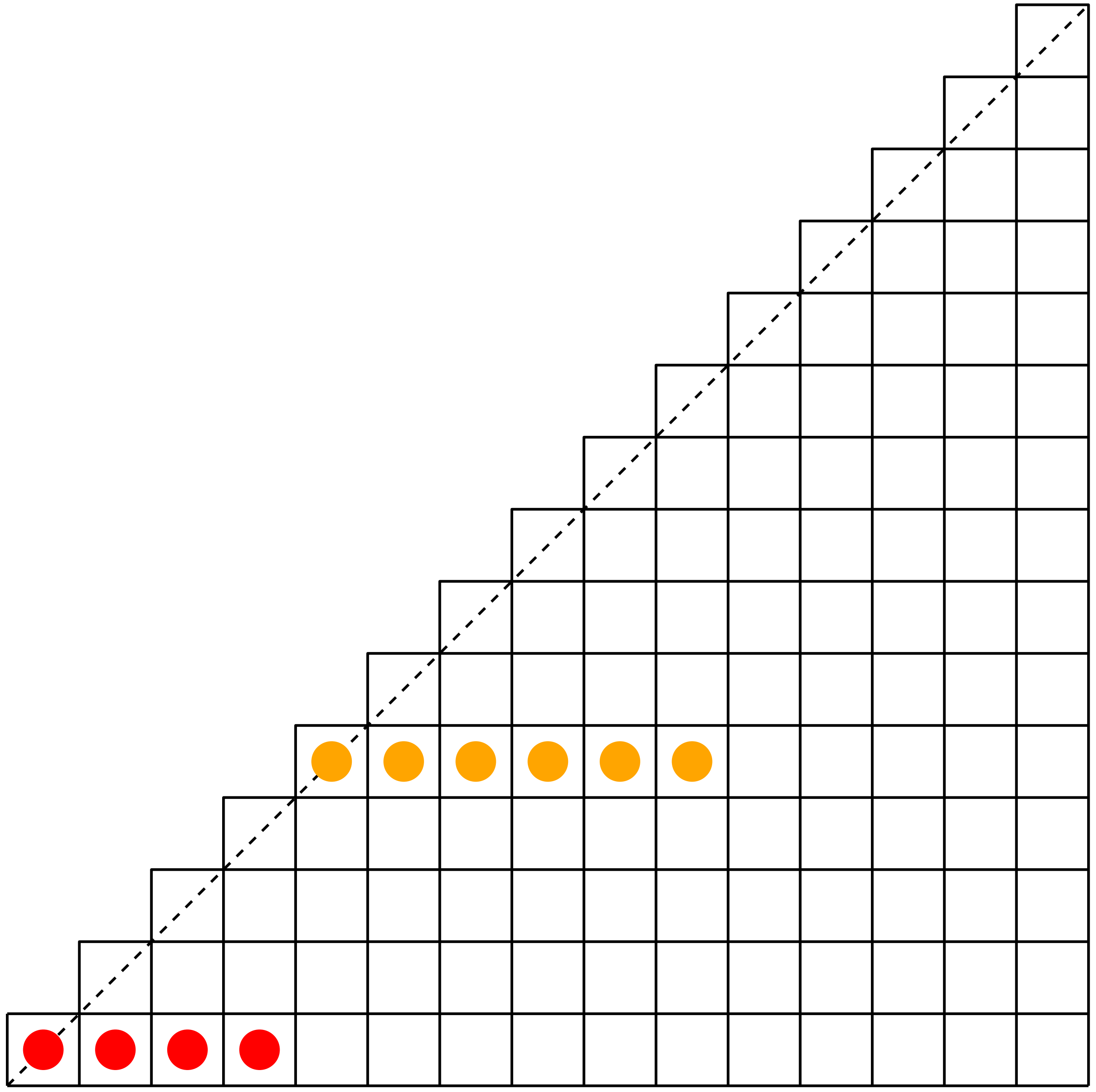

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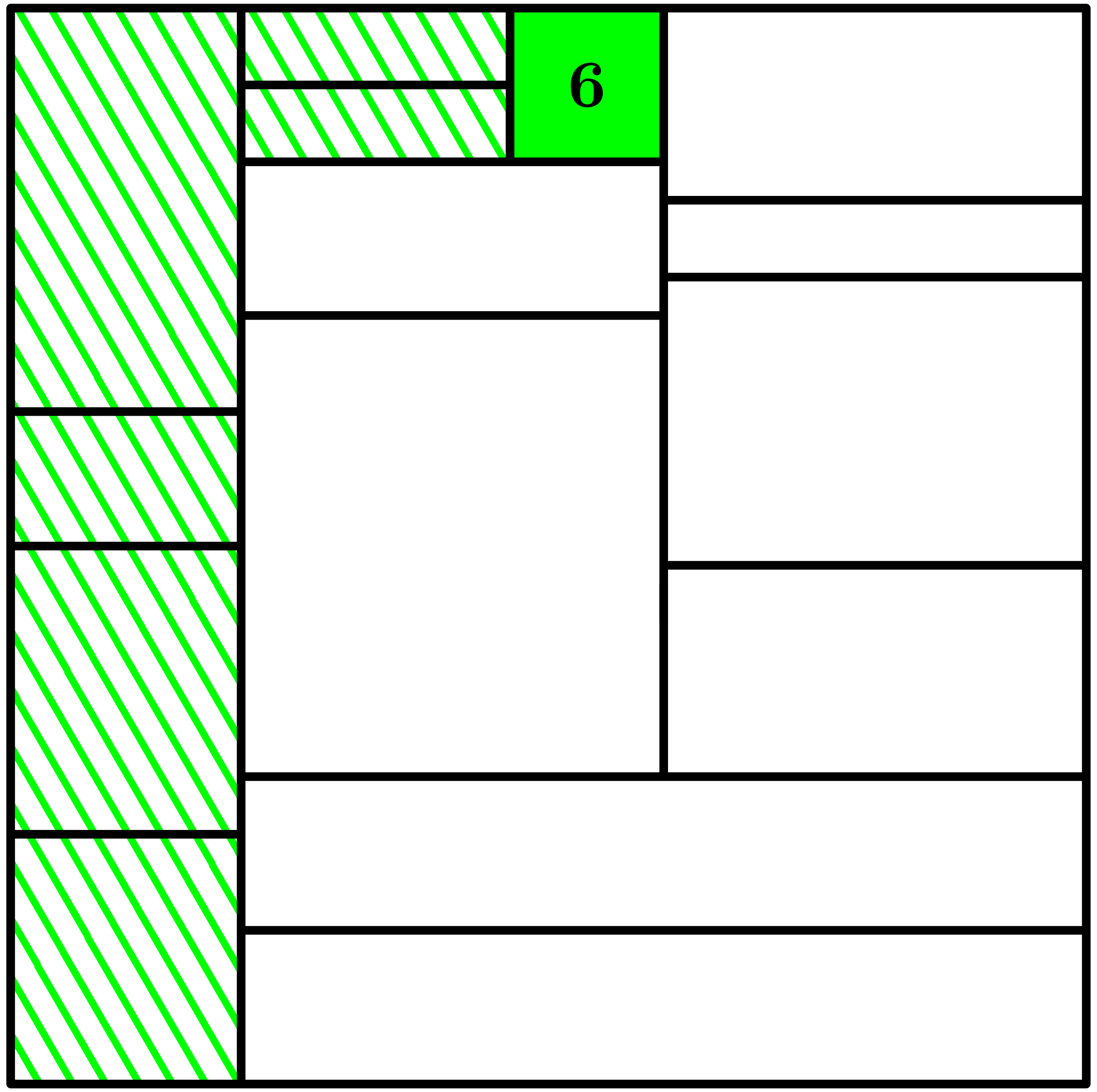

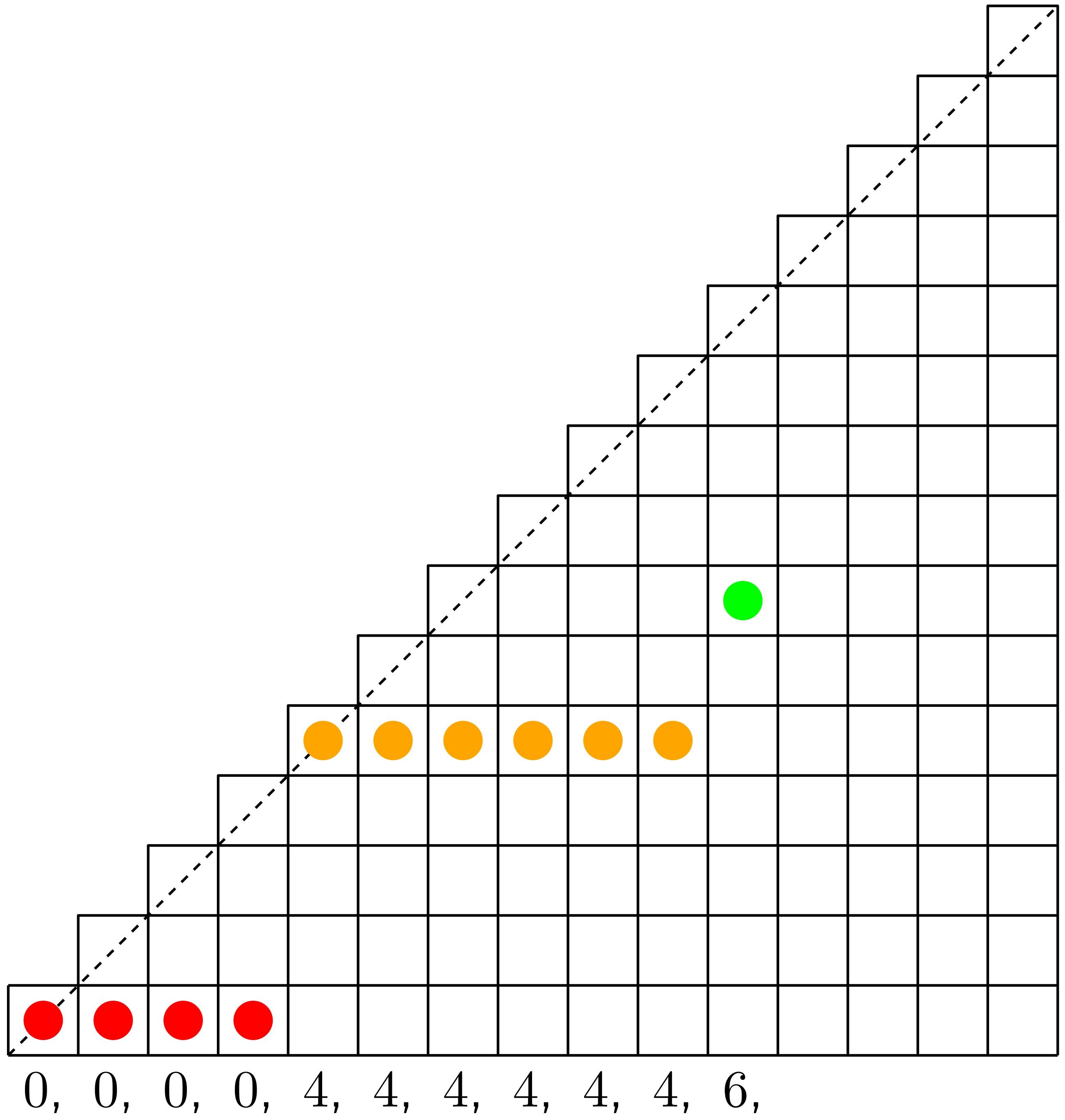

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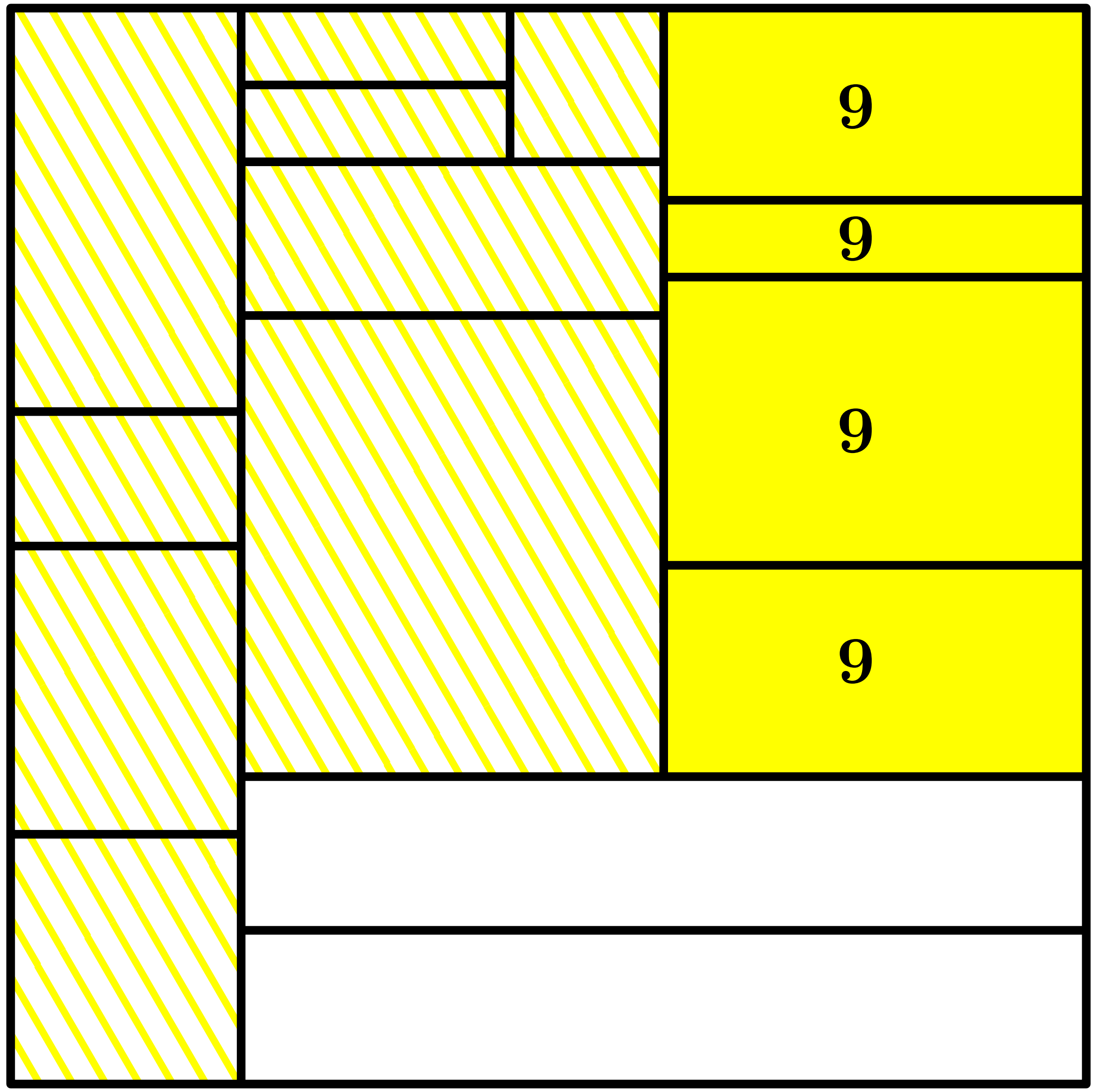
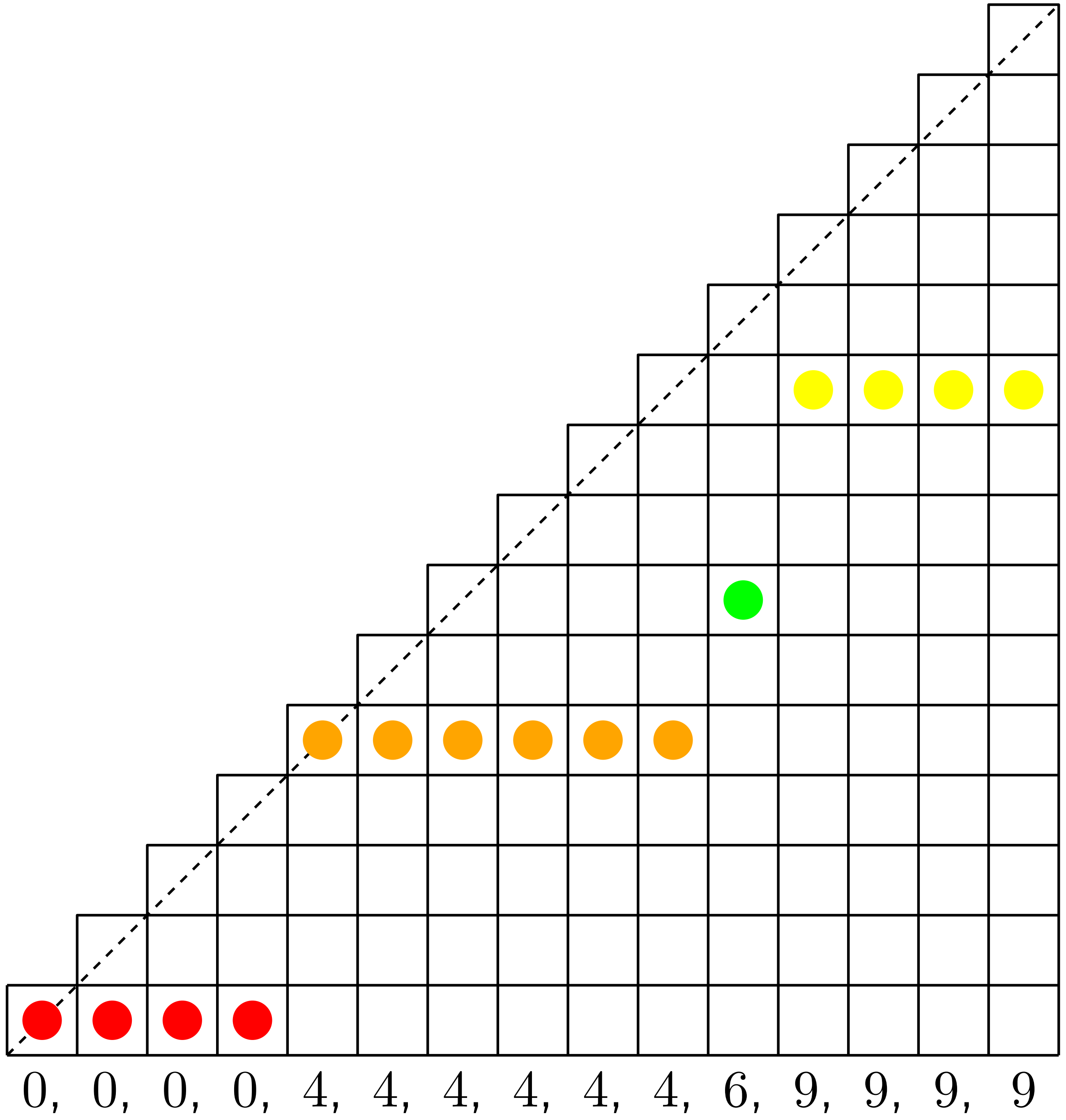
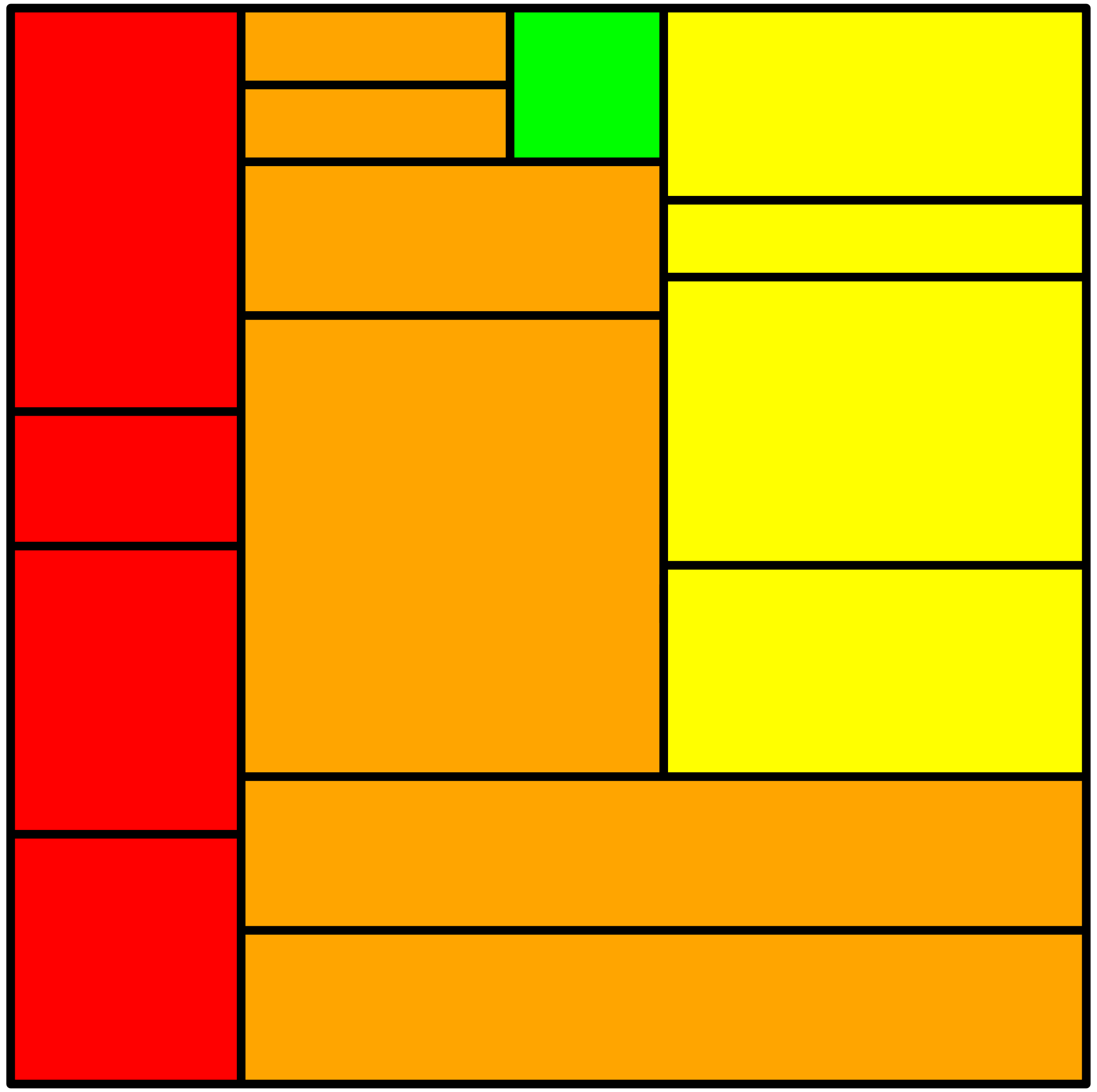
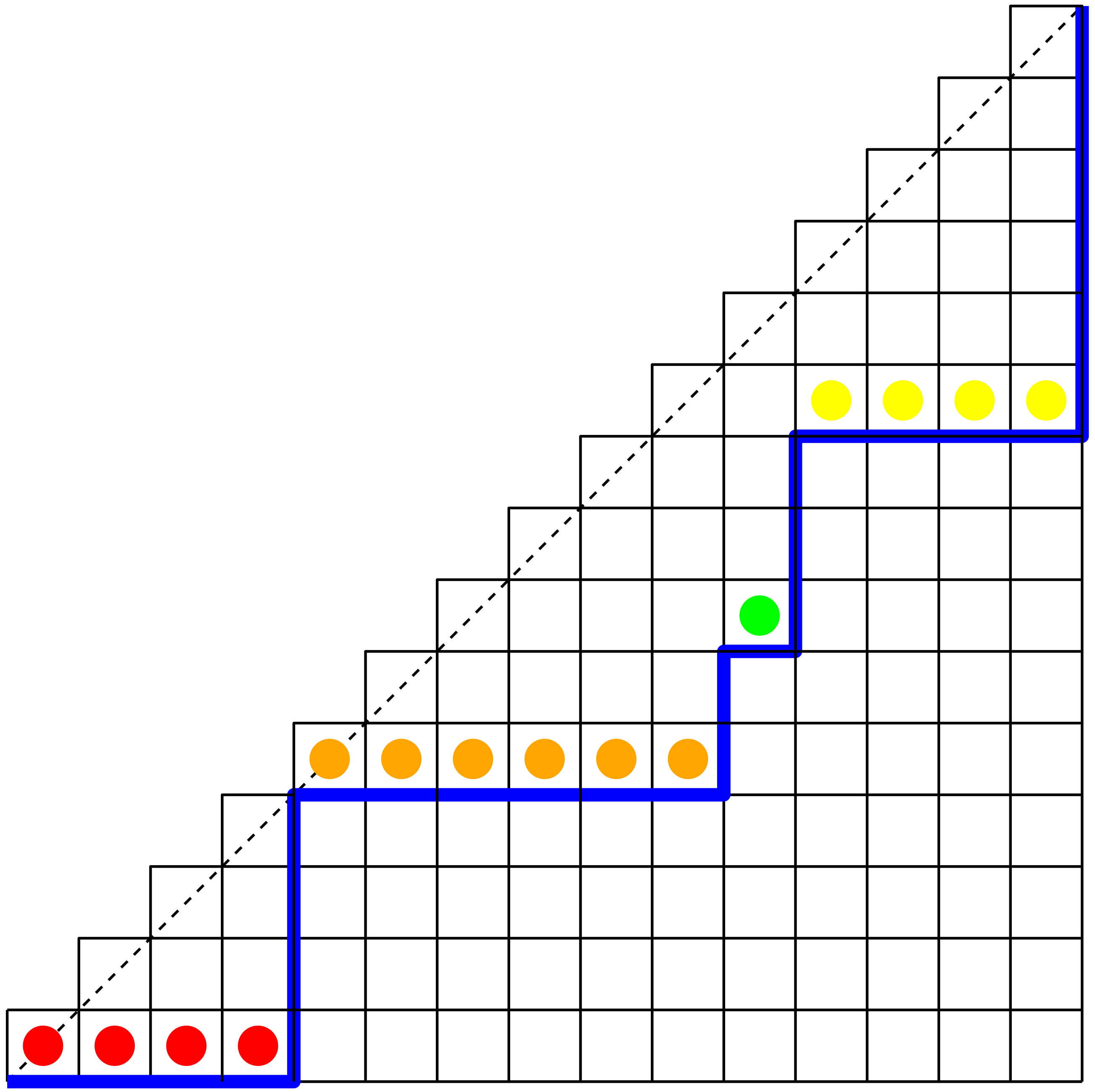




$|R_n^w(\top)| = C_n$ (Williams)

Proof: Bijection to Dyck paths via non-decreasing inversion sequences





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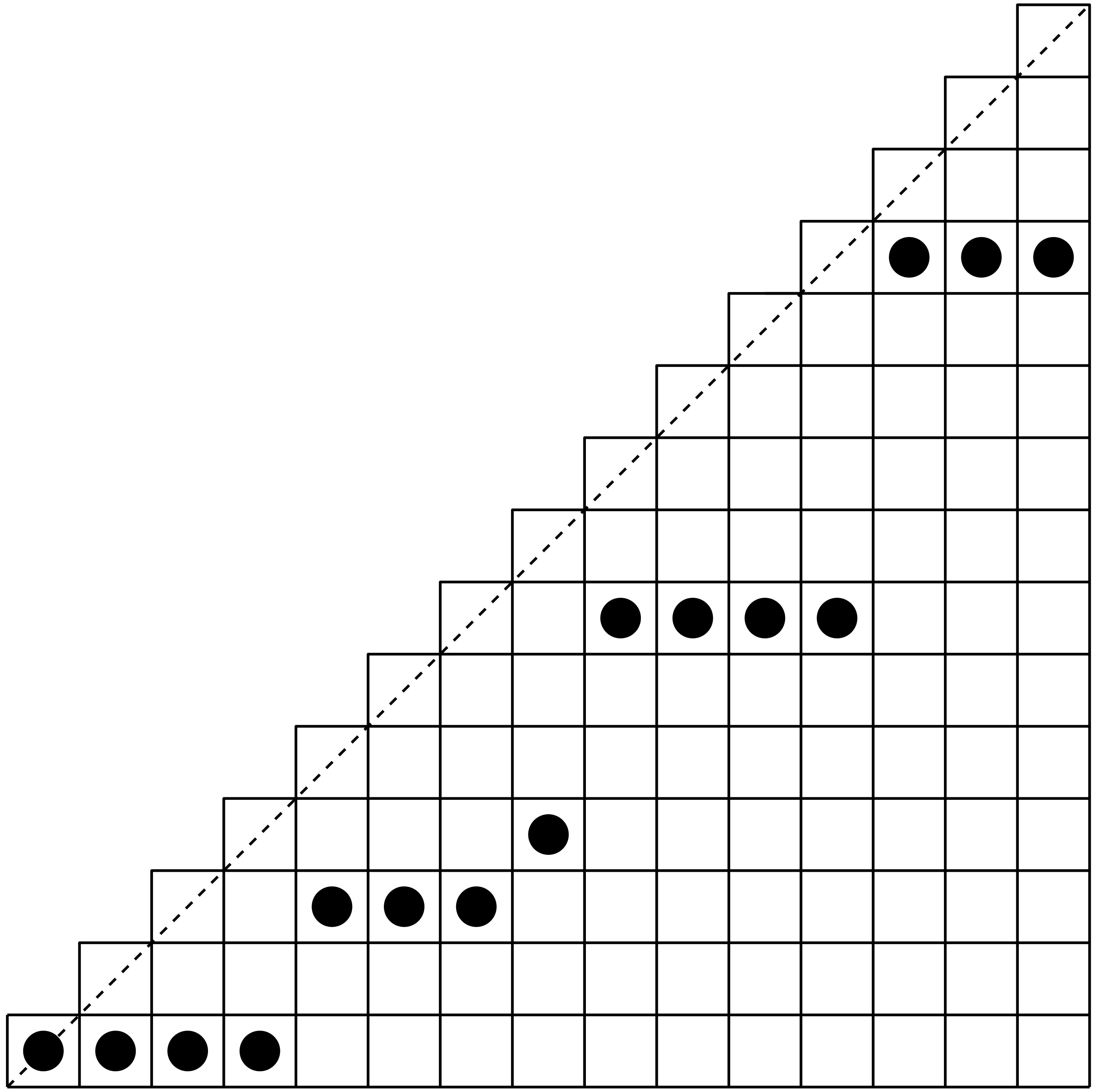
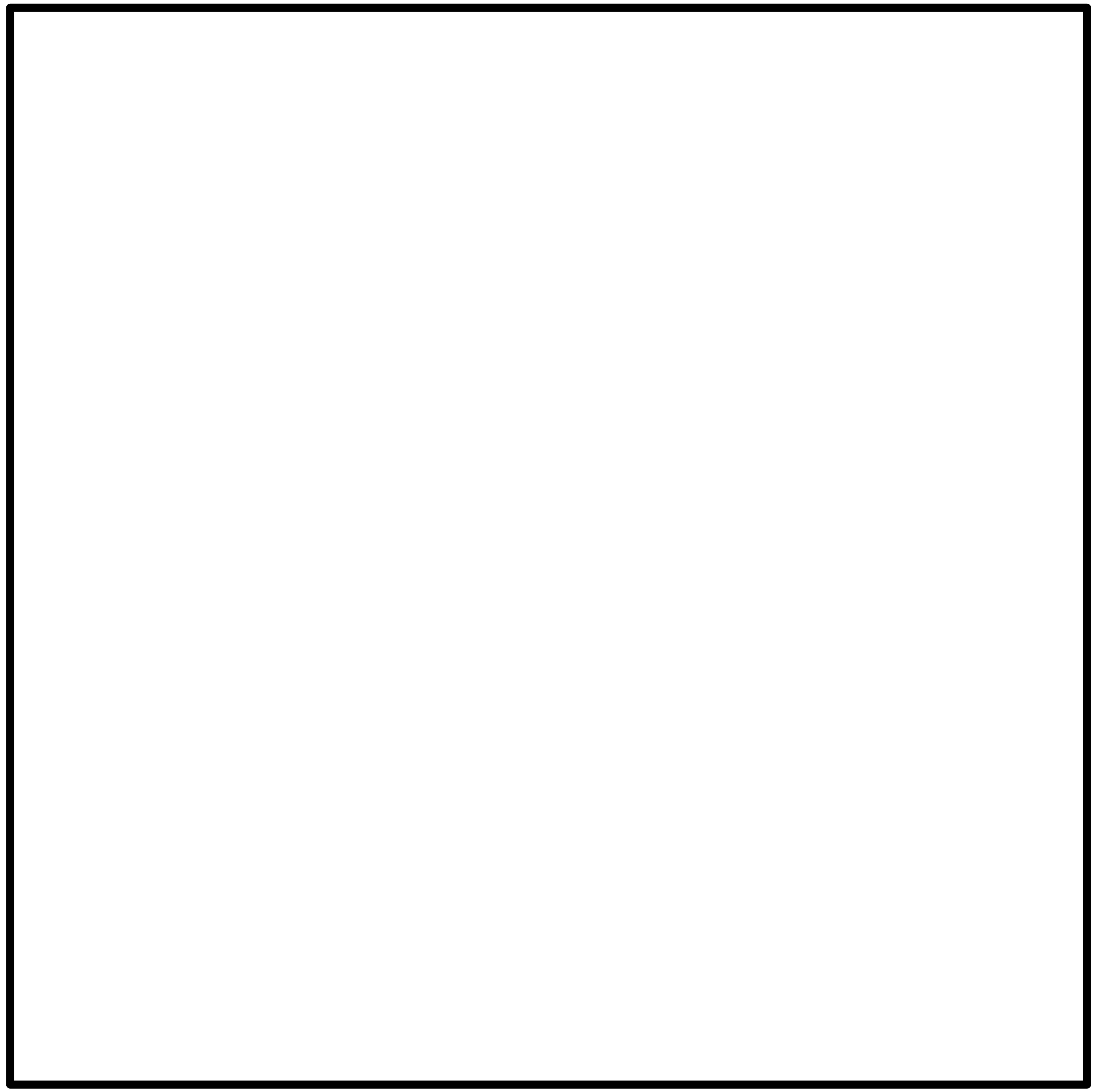


$0, 0, 0, 0, 2, 2, 2, 3, 6, 6, 6, 6, 11, 11, 11$



$|R^w_n(\top)| = C_n$ (Williams)

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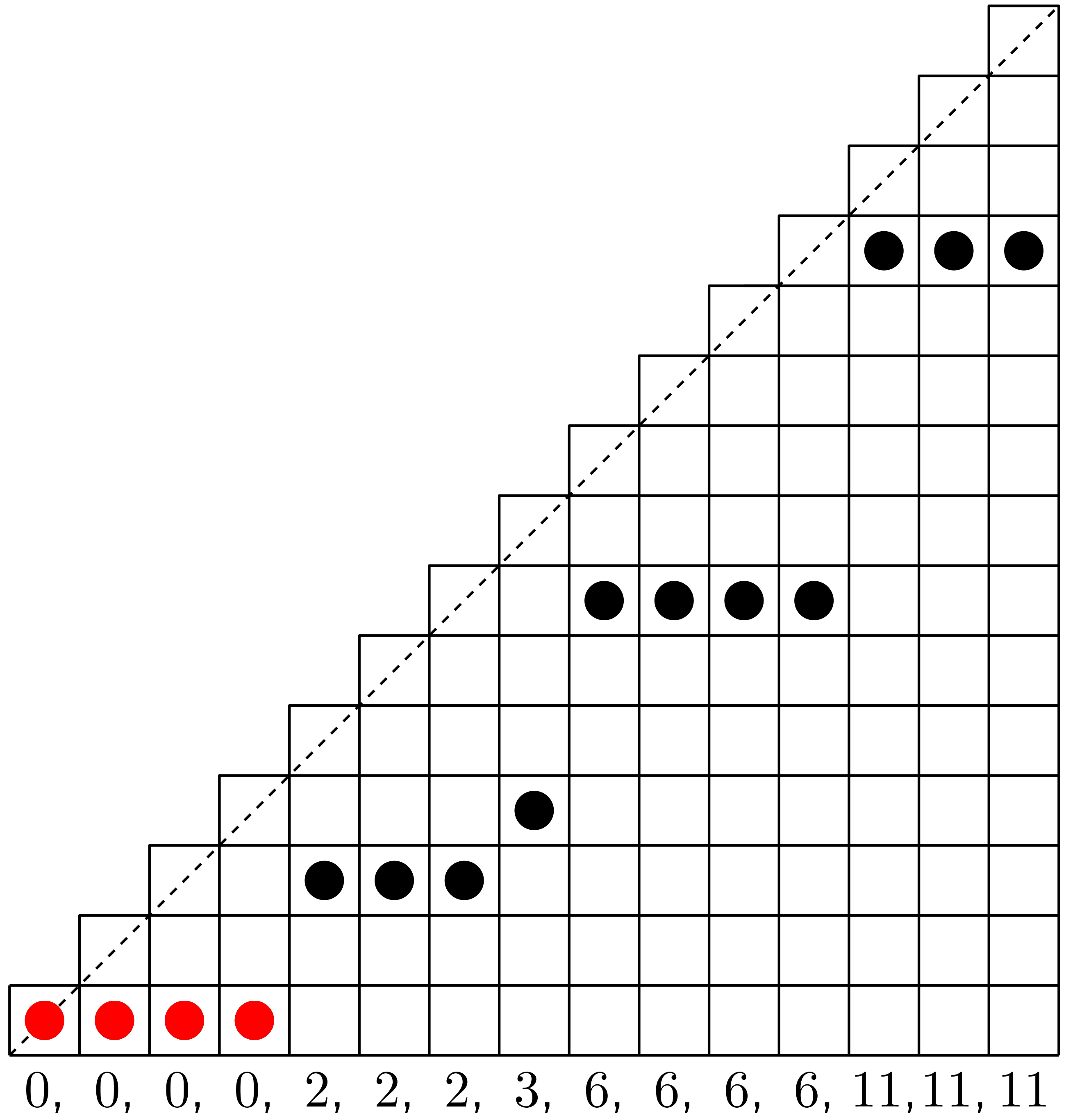

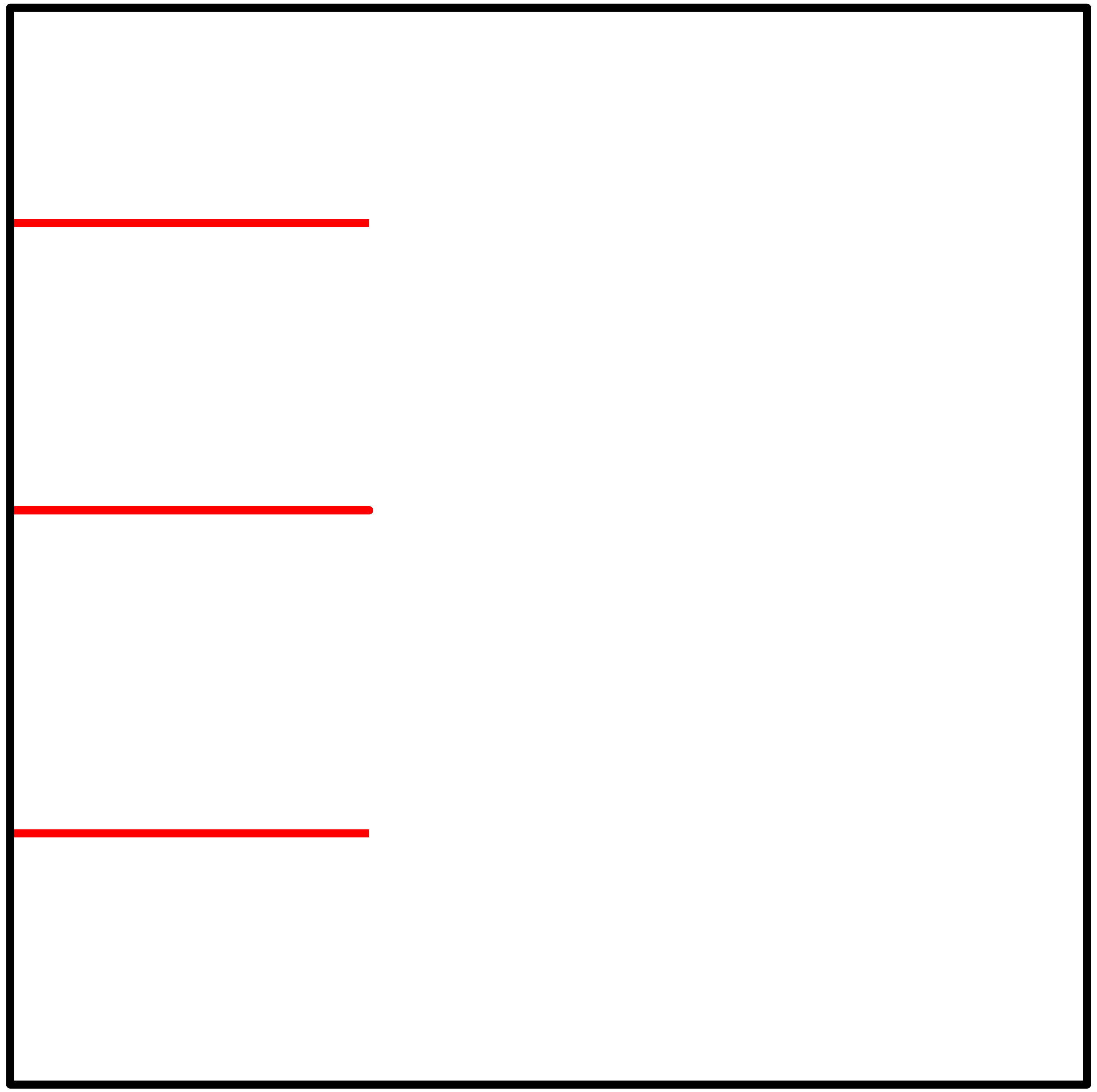

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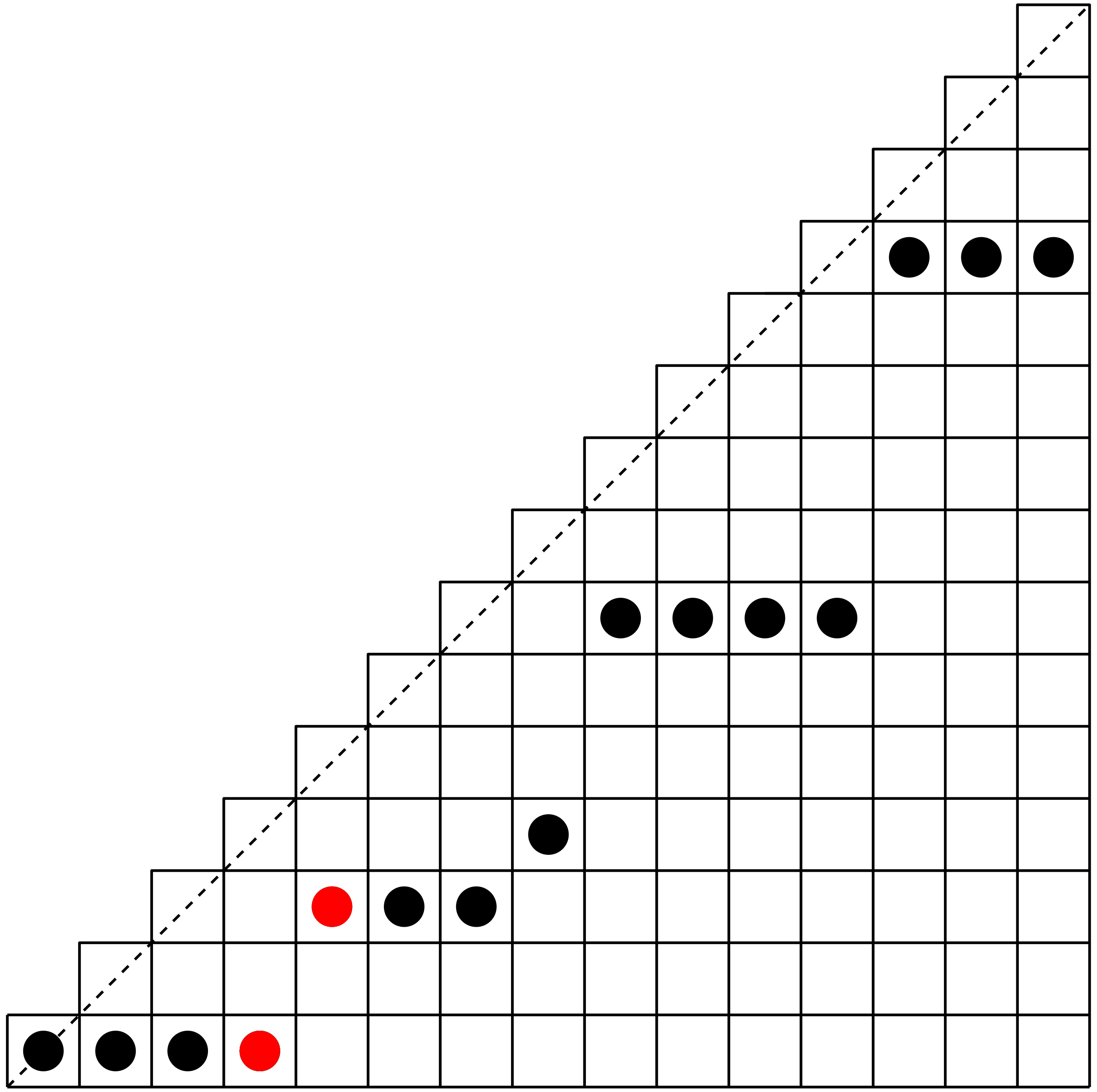
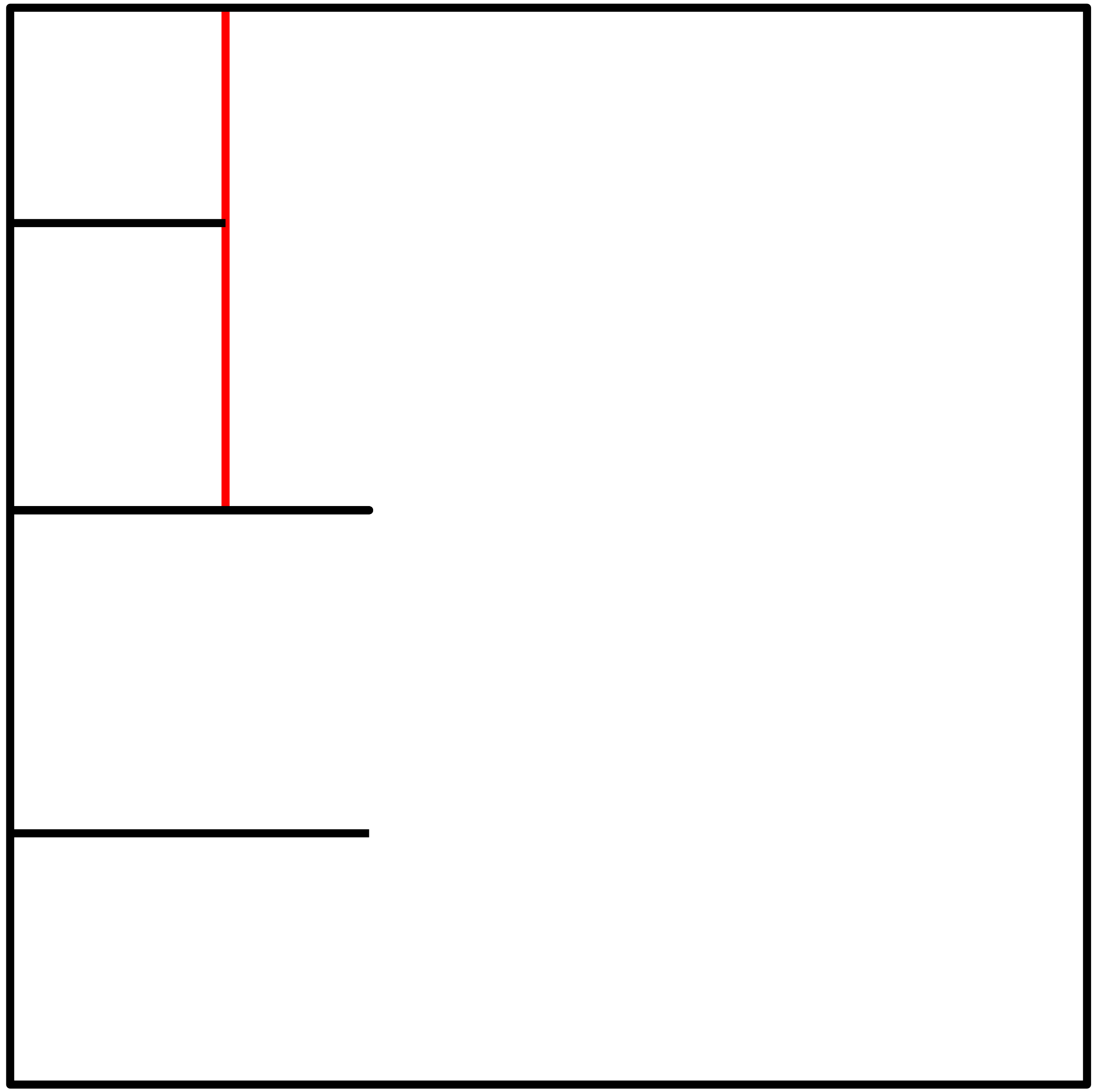


$0,\ 0,\ 0,\ 0,\ 2,\ 2,\ 2,\ 3,\ 6,\ 6,\ 6,\ 6,\ 11, 11,\ 11$



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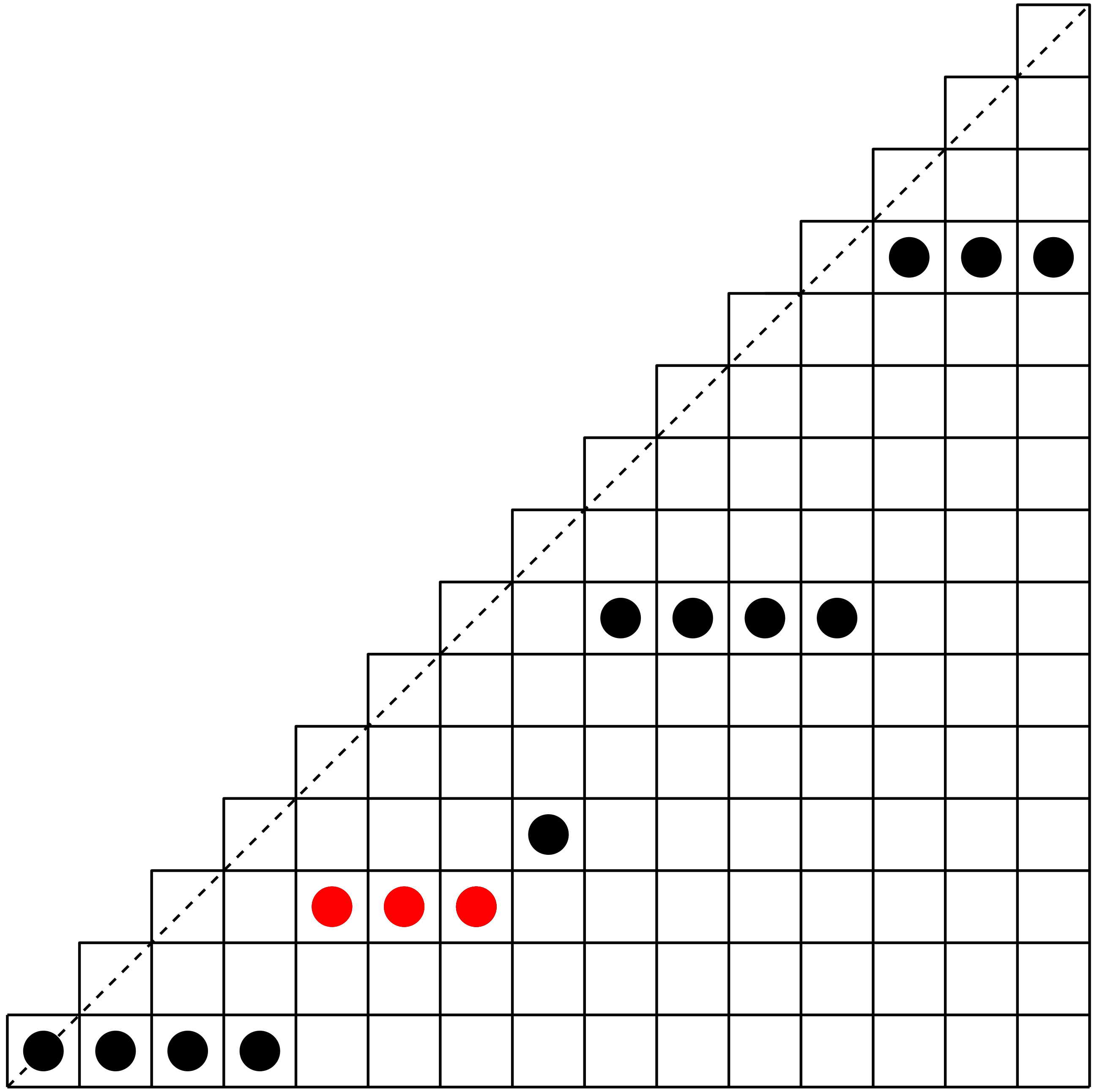
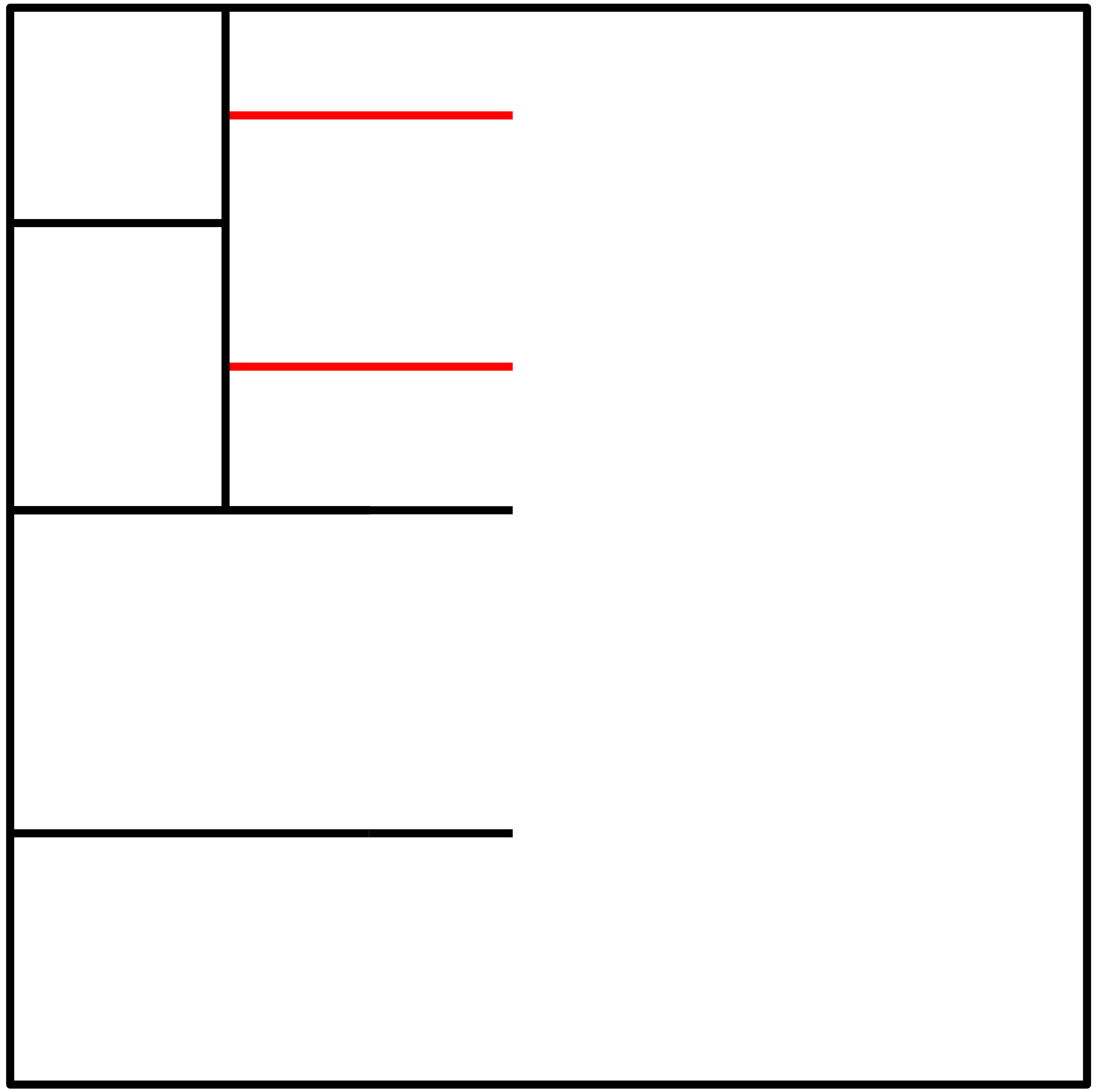


$0,\ 0,\ 0,\ 0,\ 2,\ 2,\ 2,\ 3,\ 6,\ 6,\ 6,\ 6,\ 11, 11, 11$



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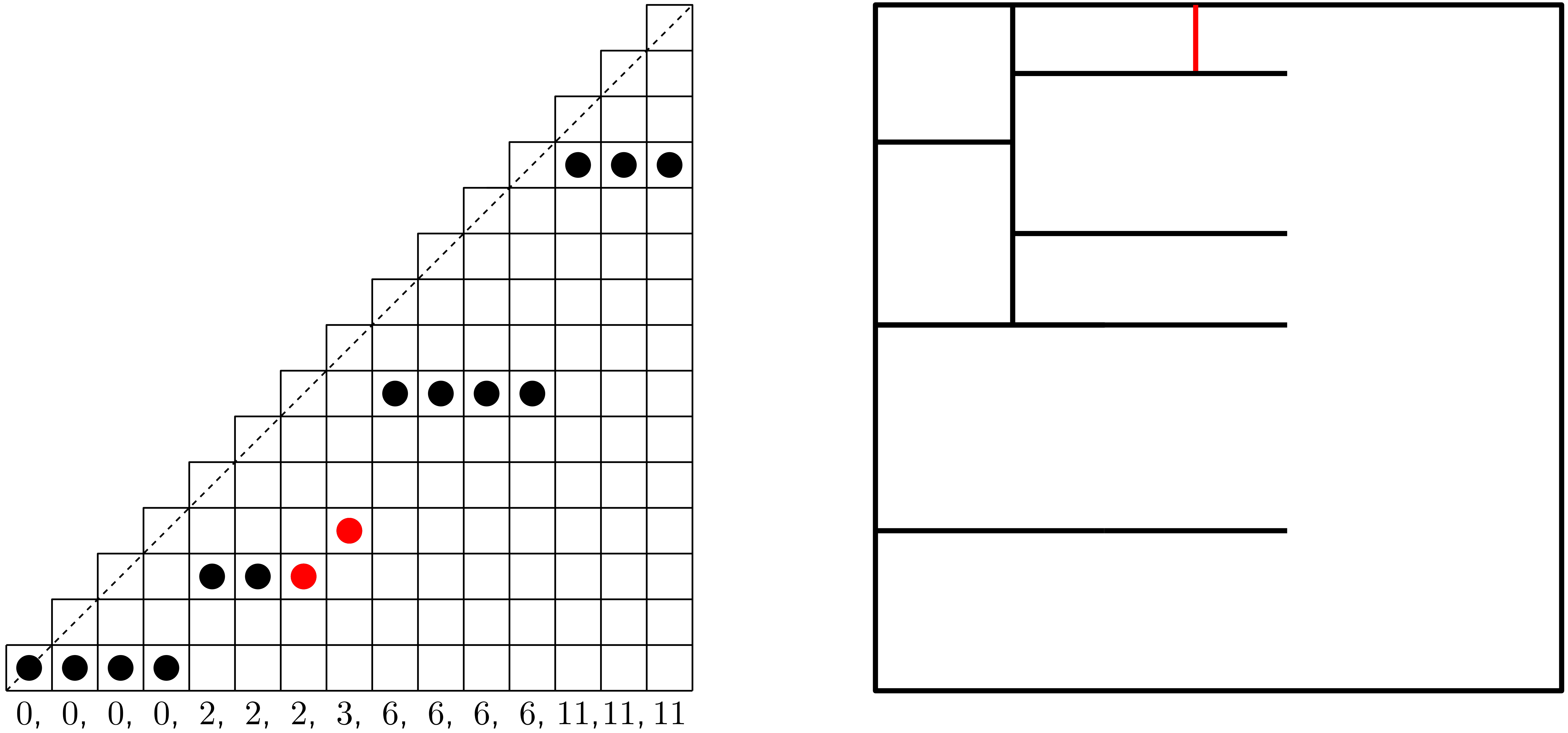

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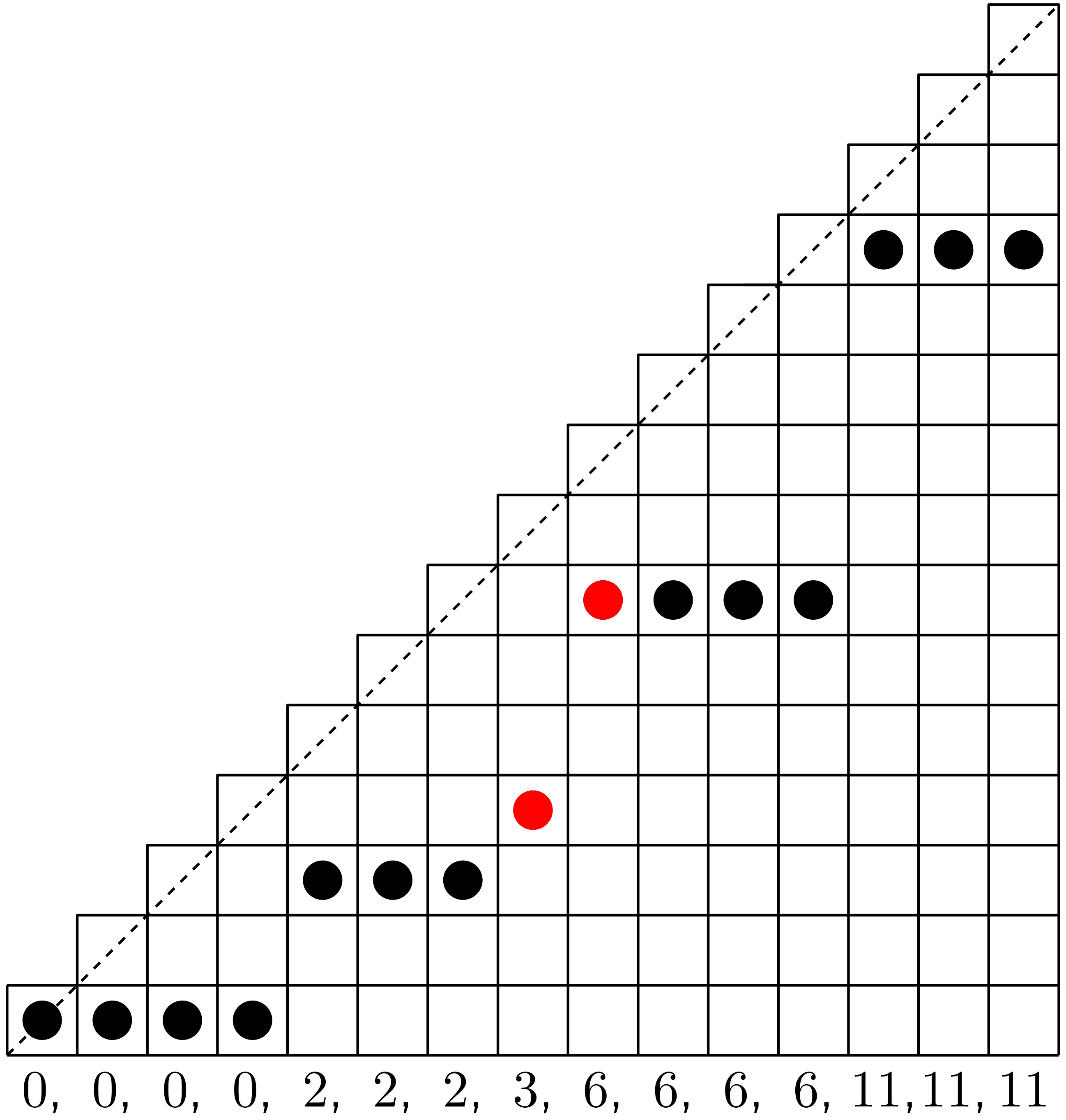

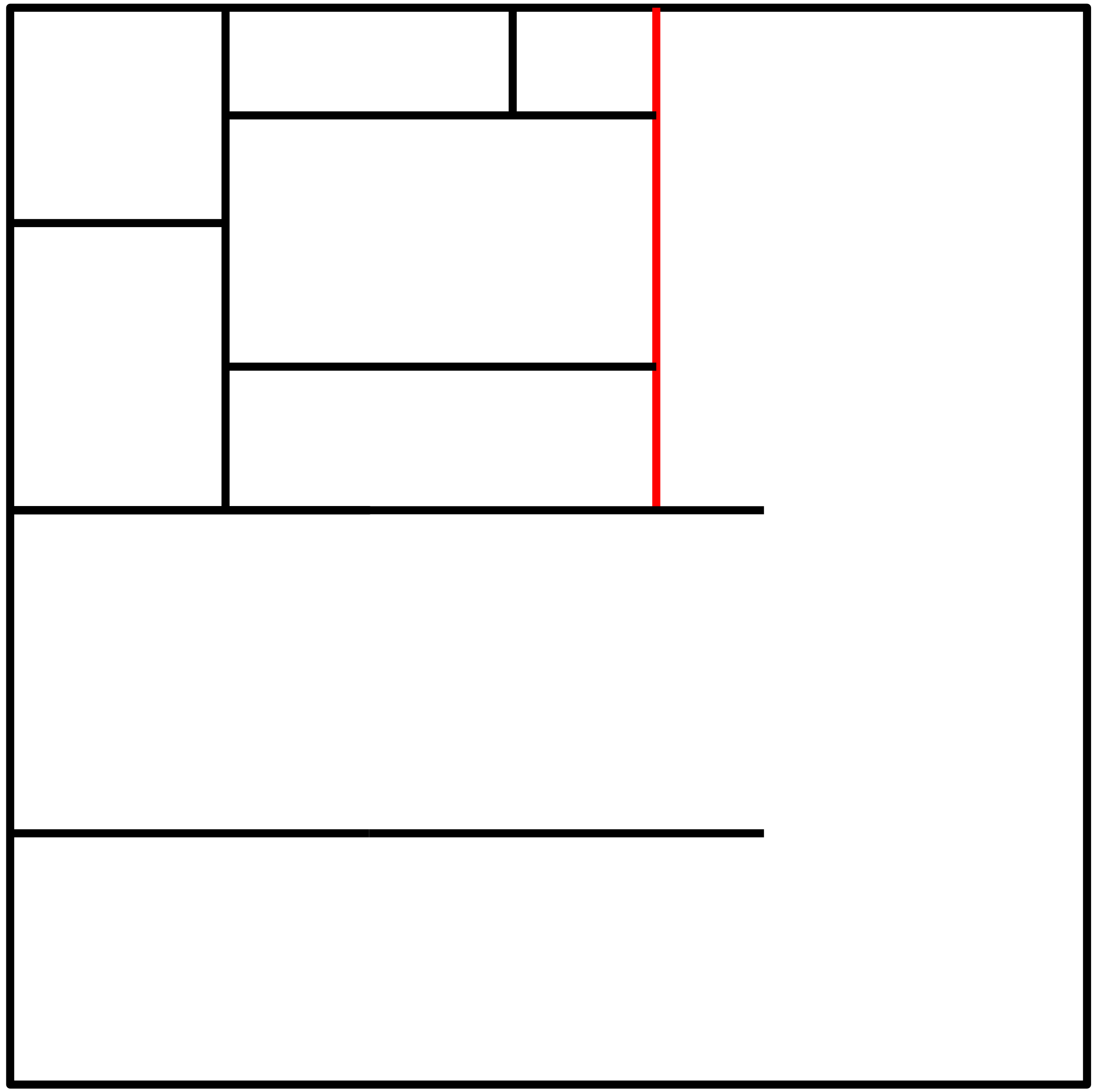

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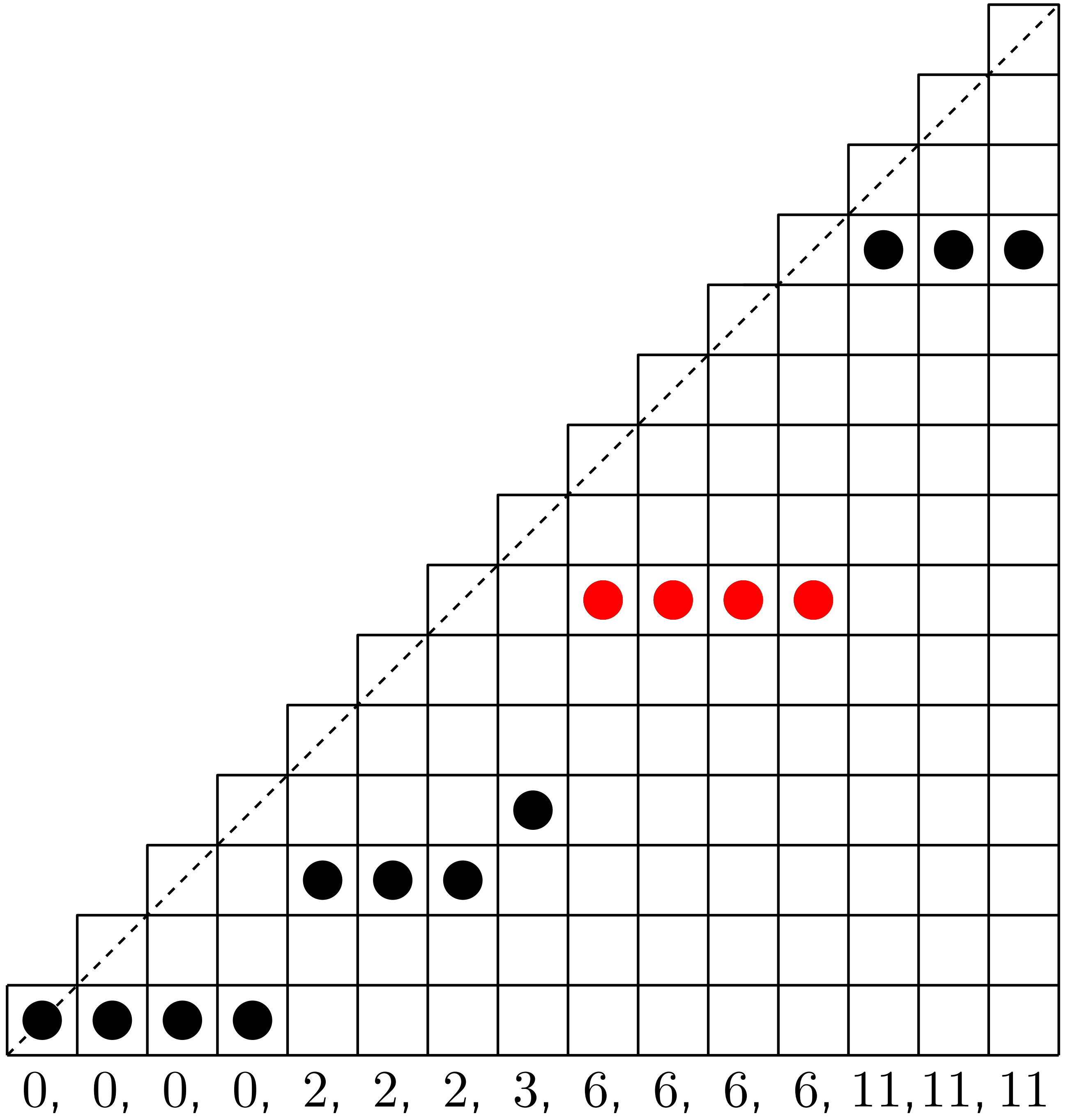

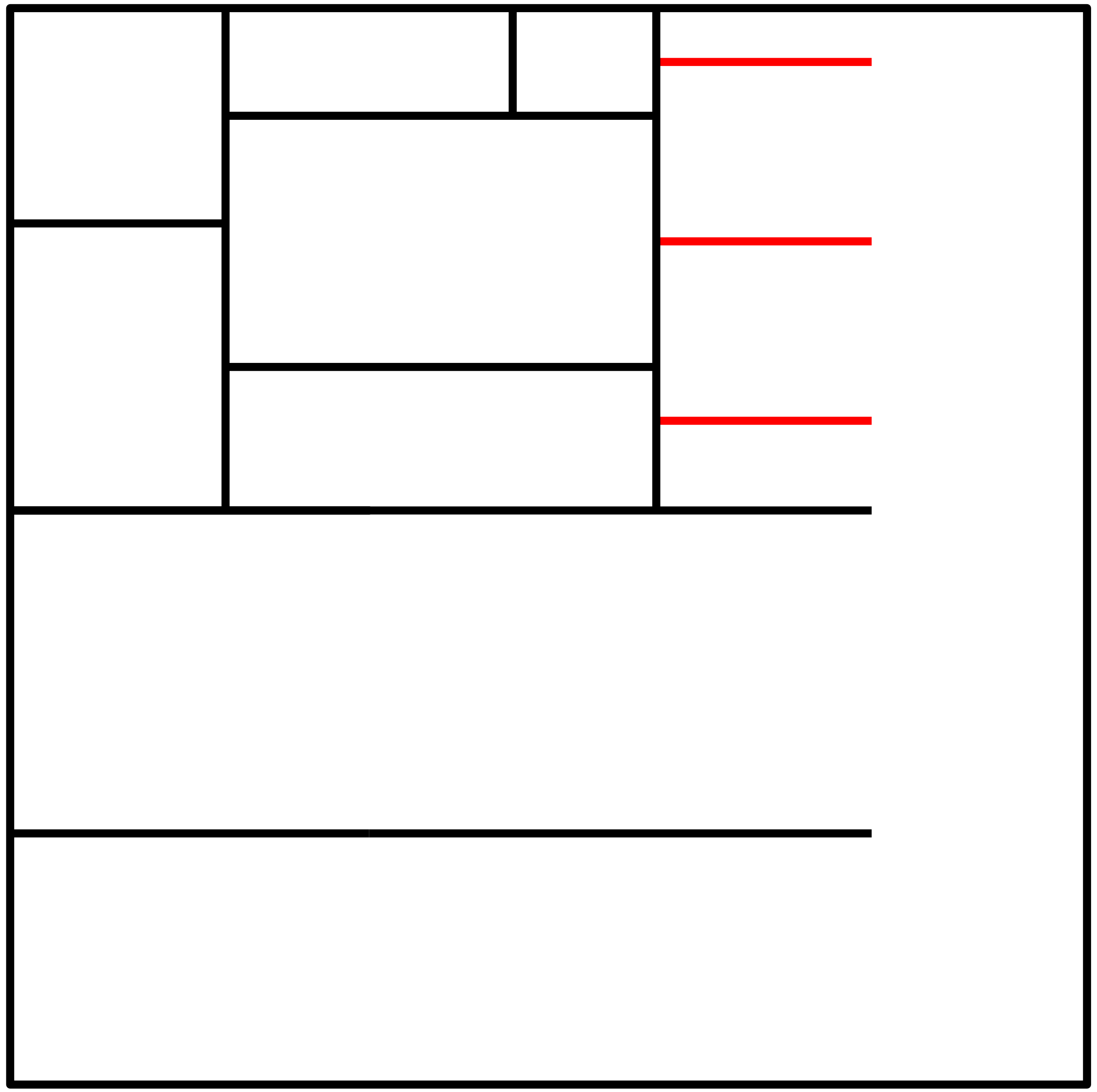

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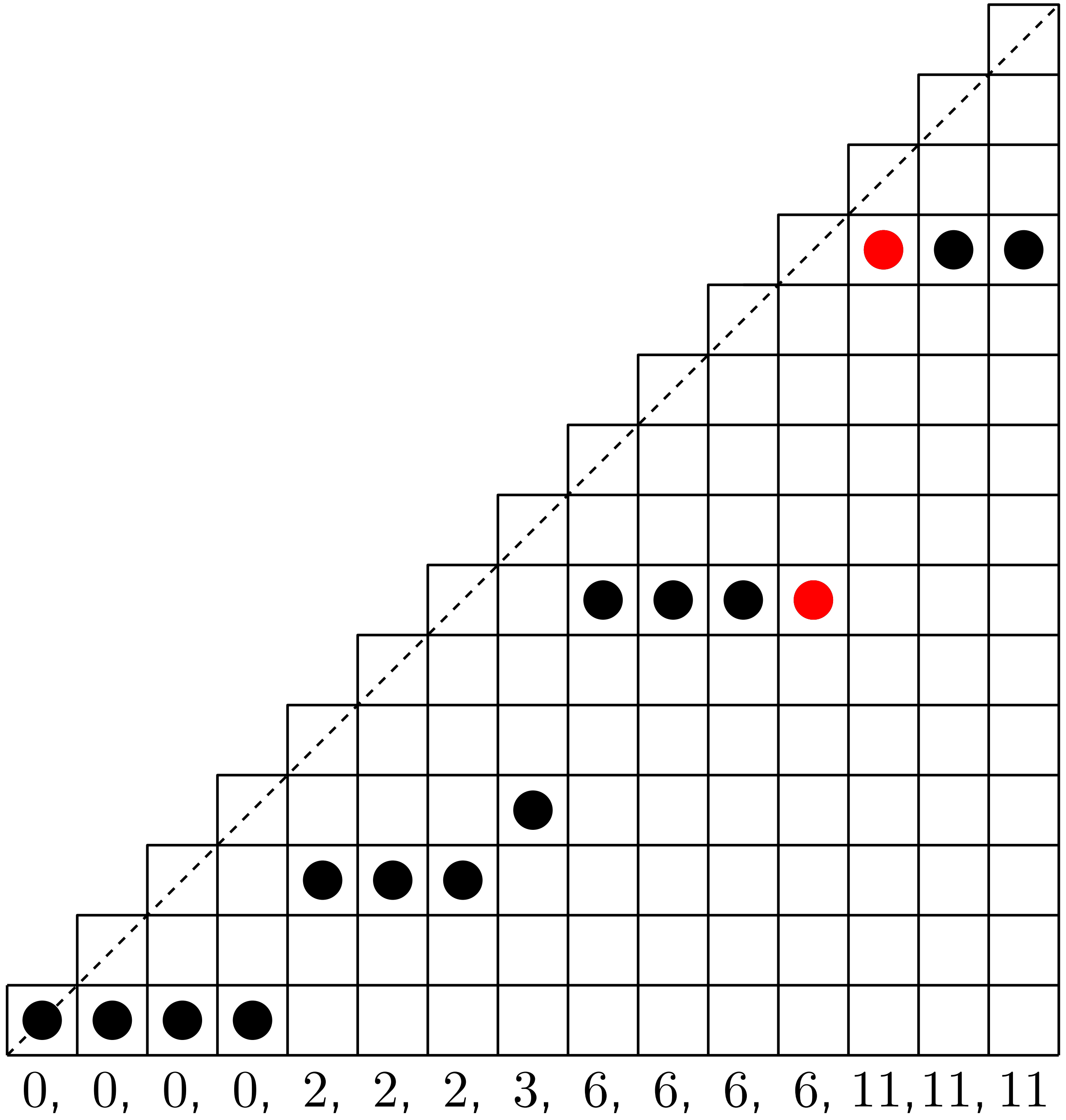

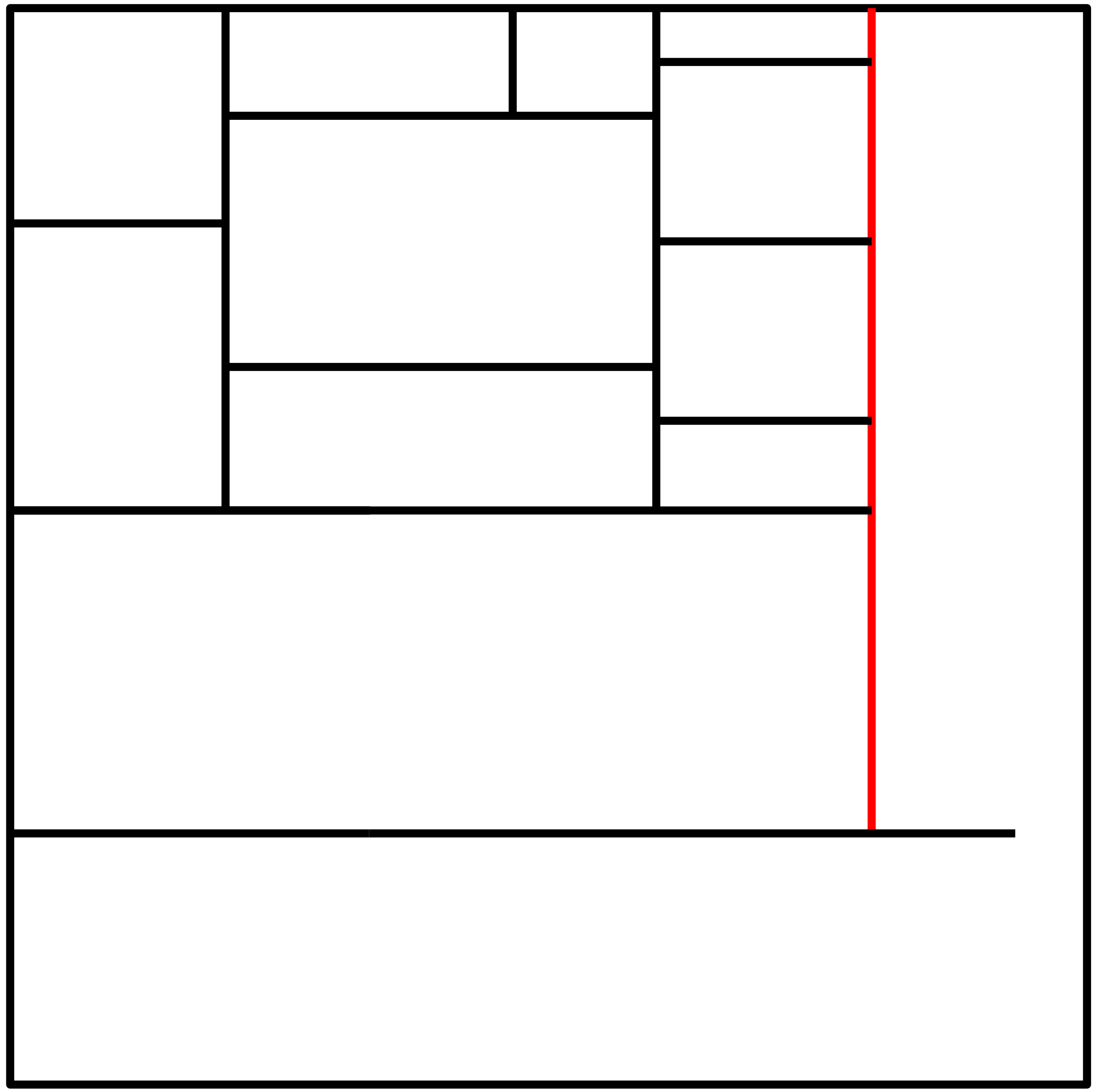

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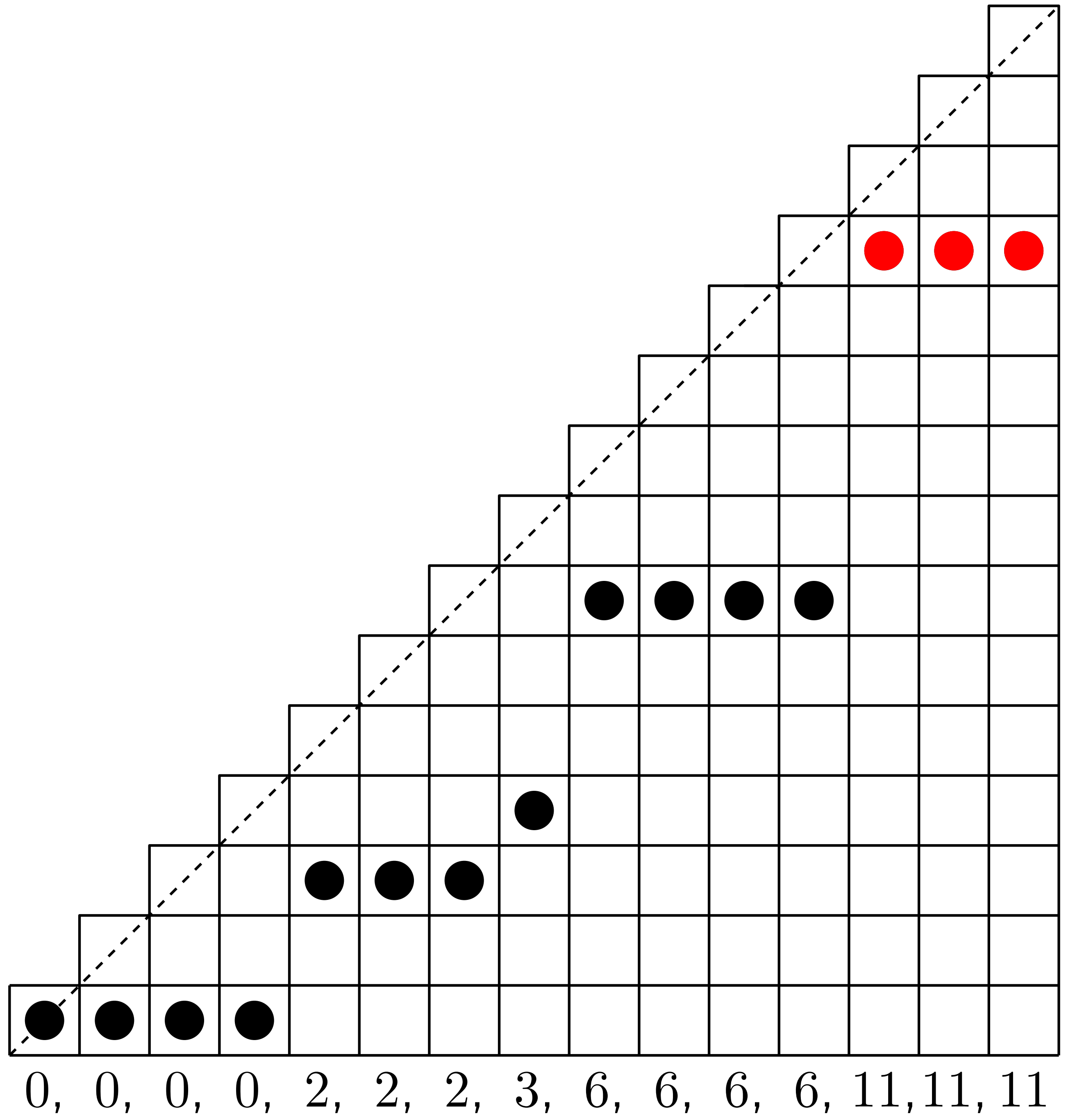

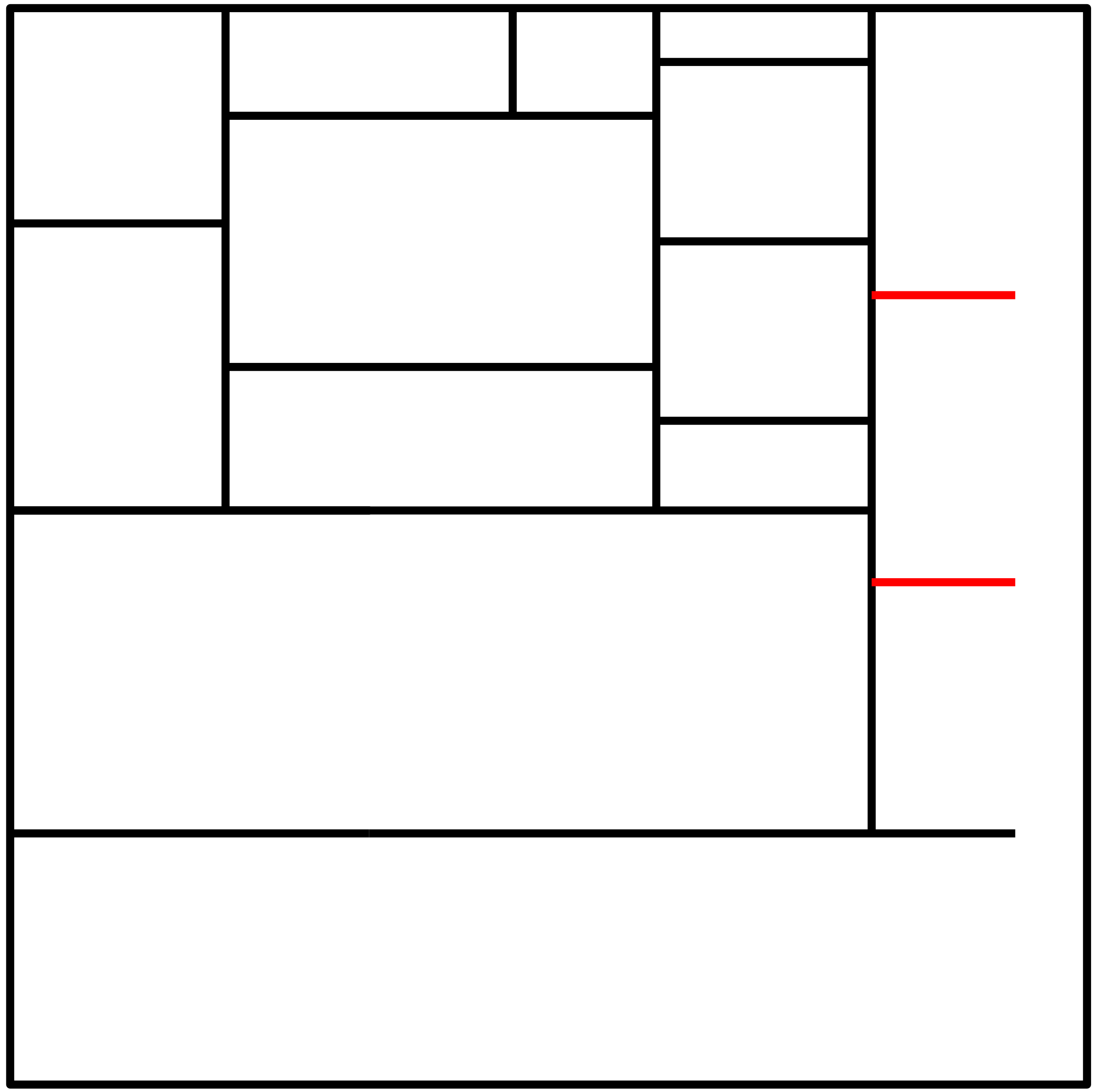

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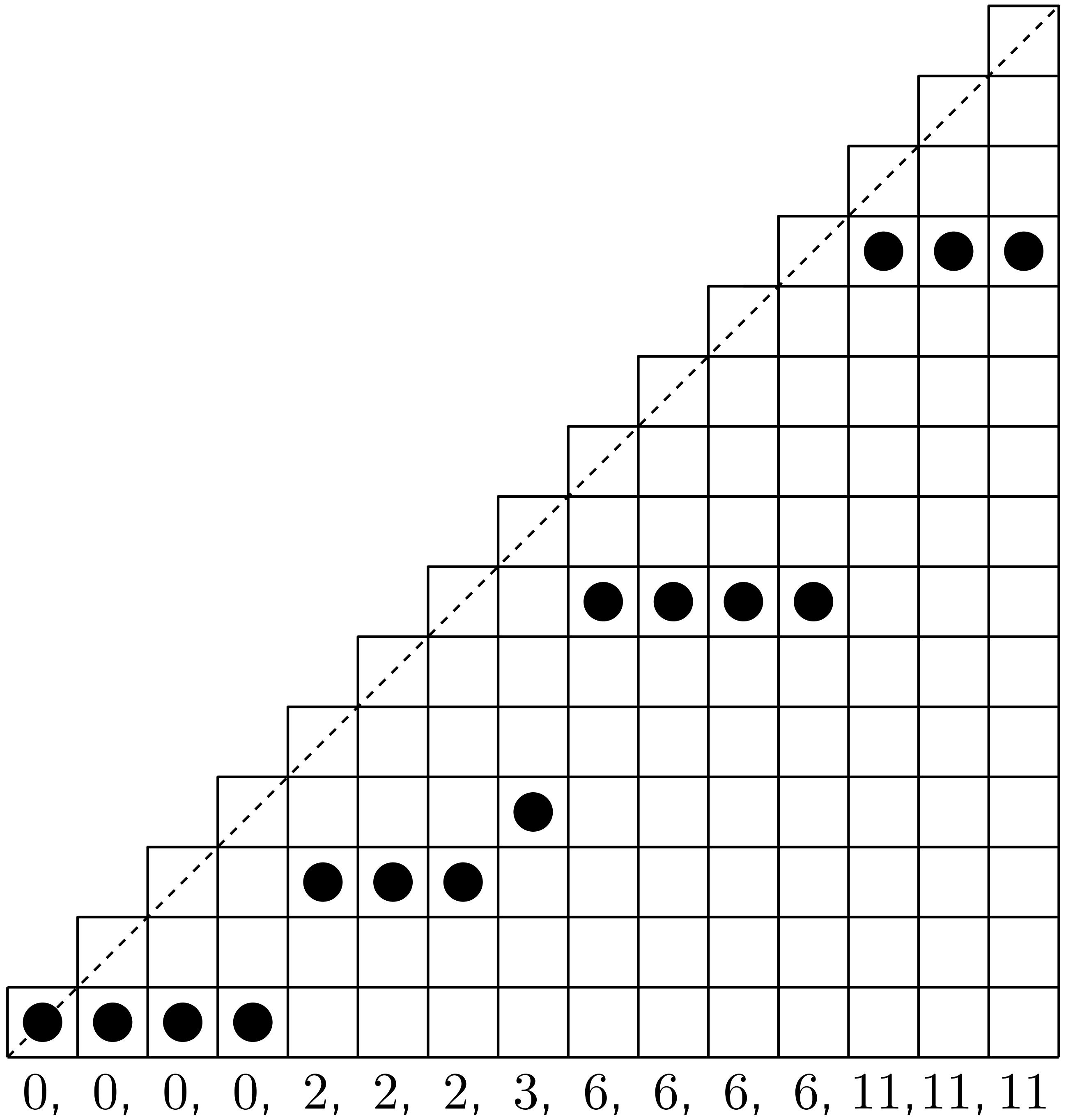
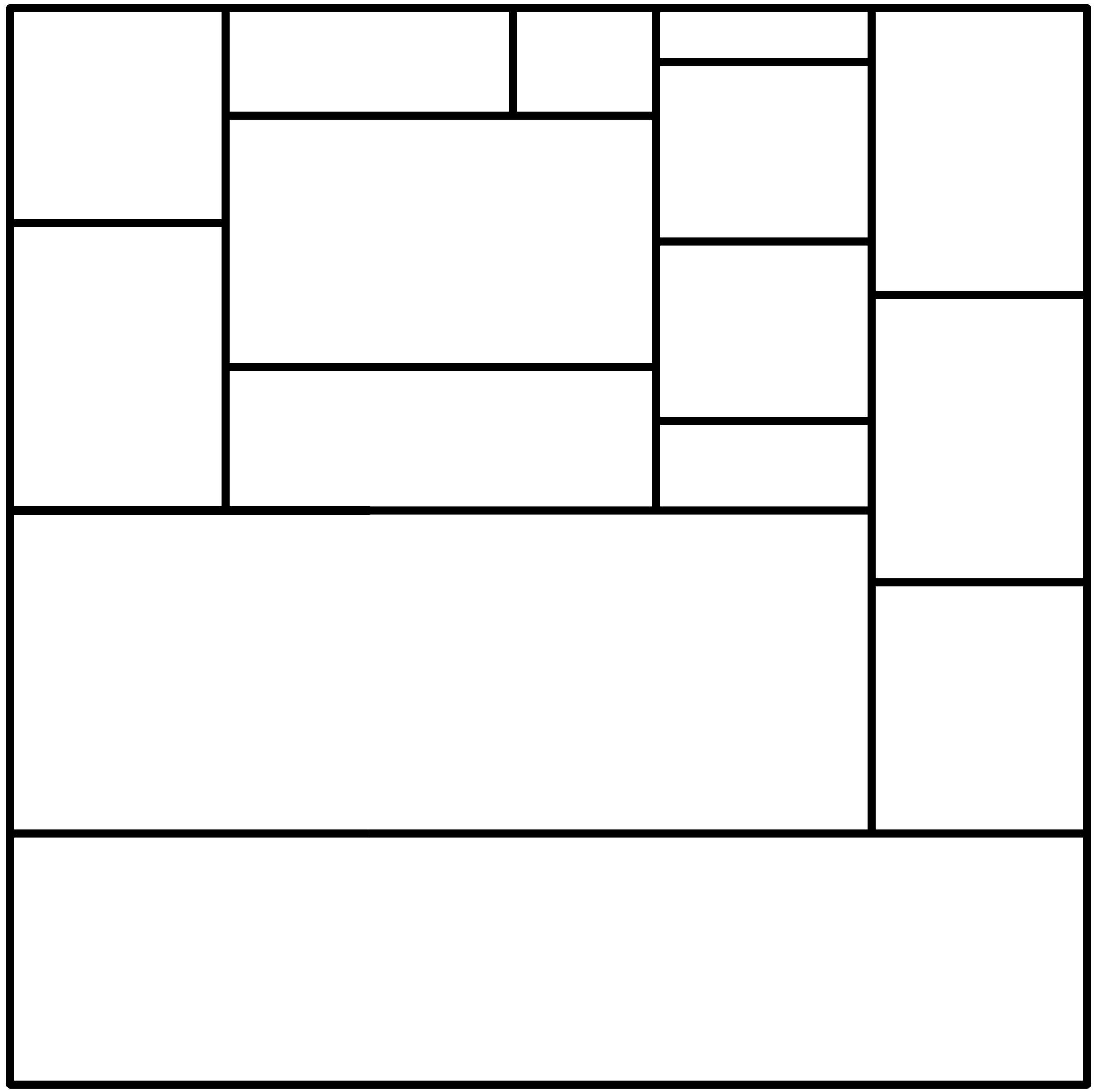
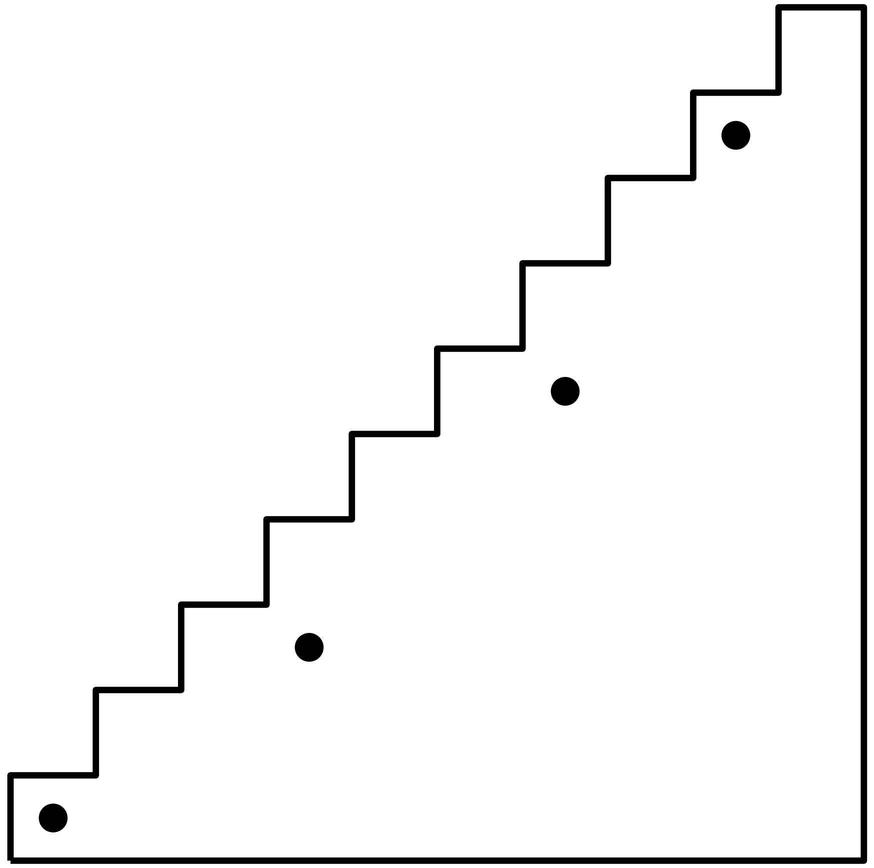


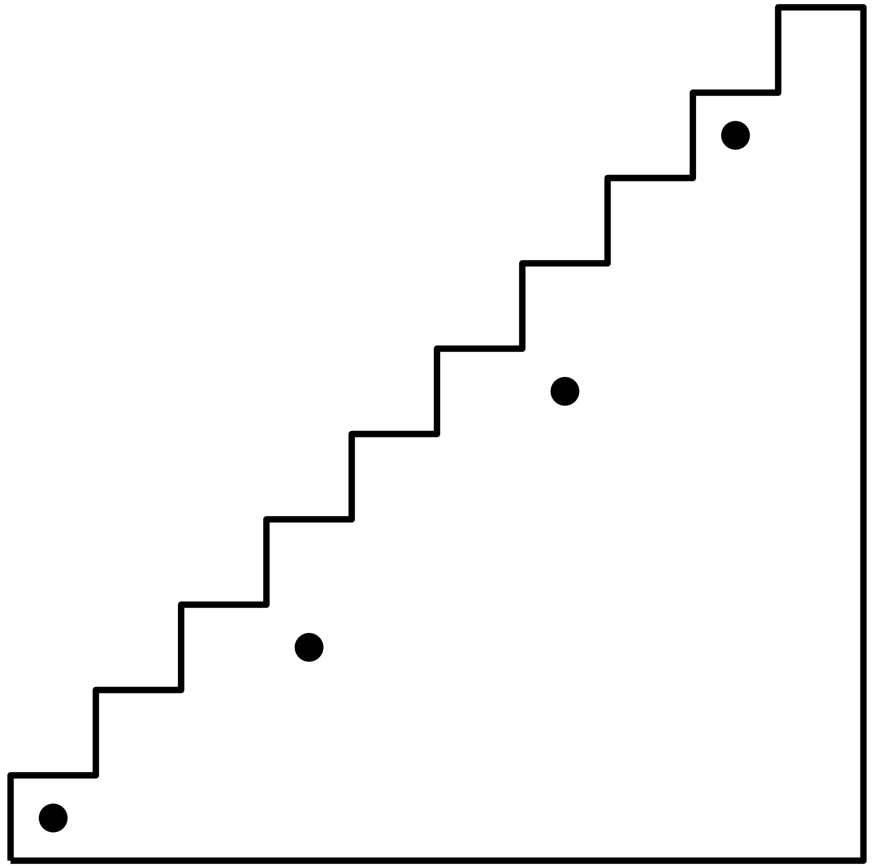


$|R_n^s(\top)| = |I_n(010, 101, 120, 201)|$, OEIS A279555 (Asinowski and P)

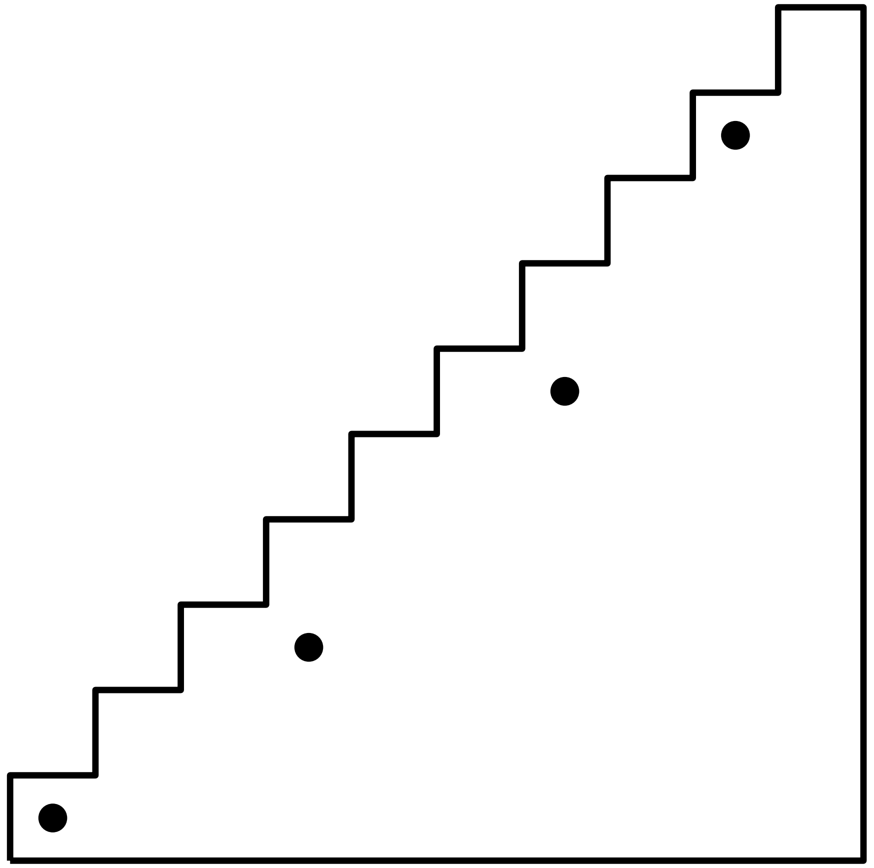
$I_n(010, 101, 120, 201)$



$I_n(010, 110, 120, 210)$



$I_n(010, 100, 120, 210)$



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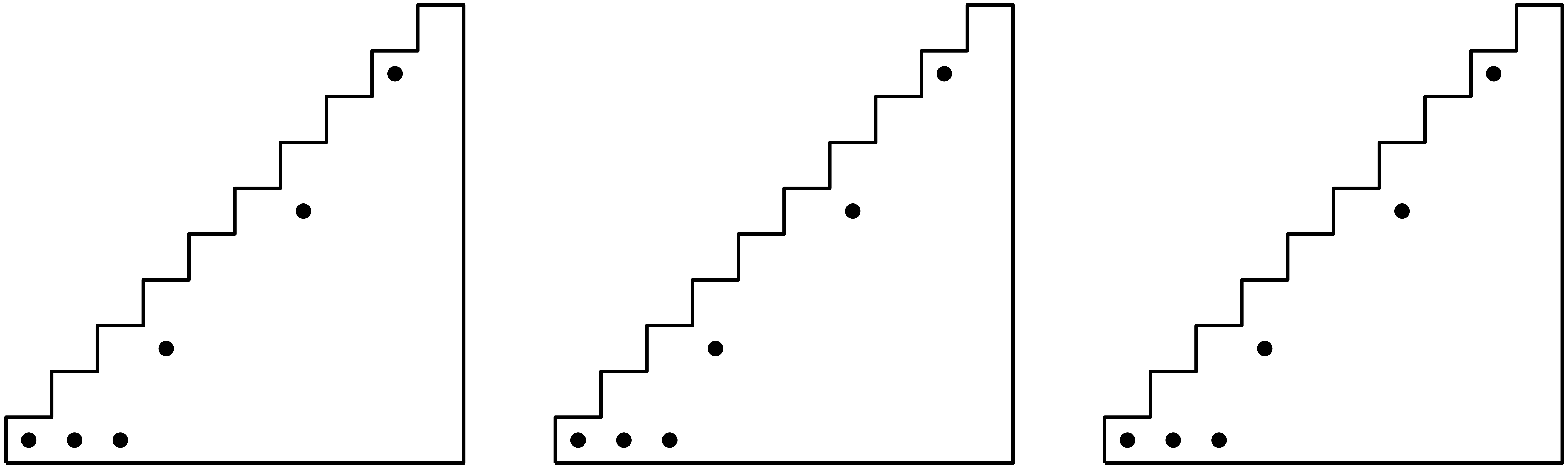
$I_n(010, 101, 120, 201)$ $I_n(010, 110, 120, 210)$ $I_n(010, 100, 120, 210)$



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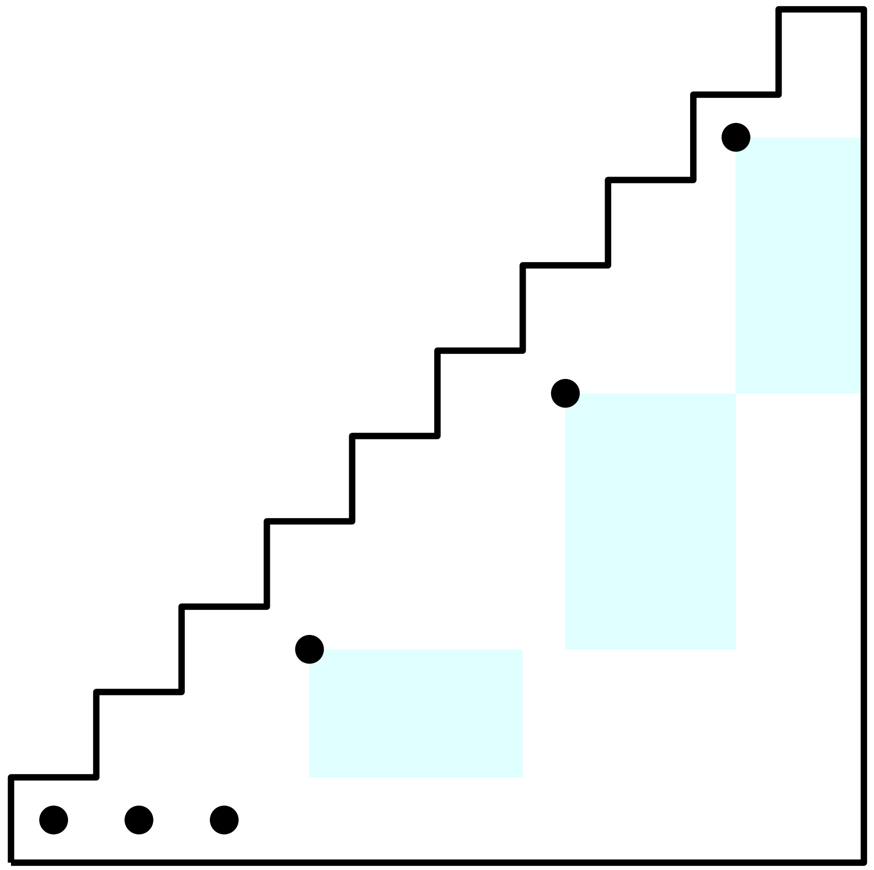
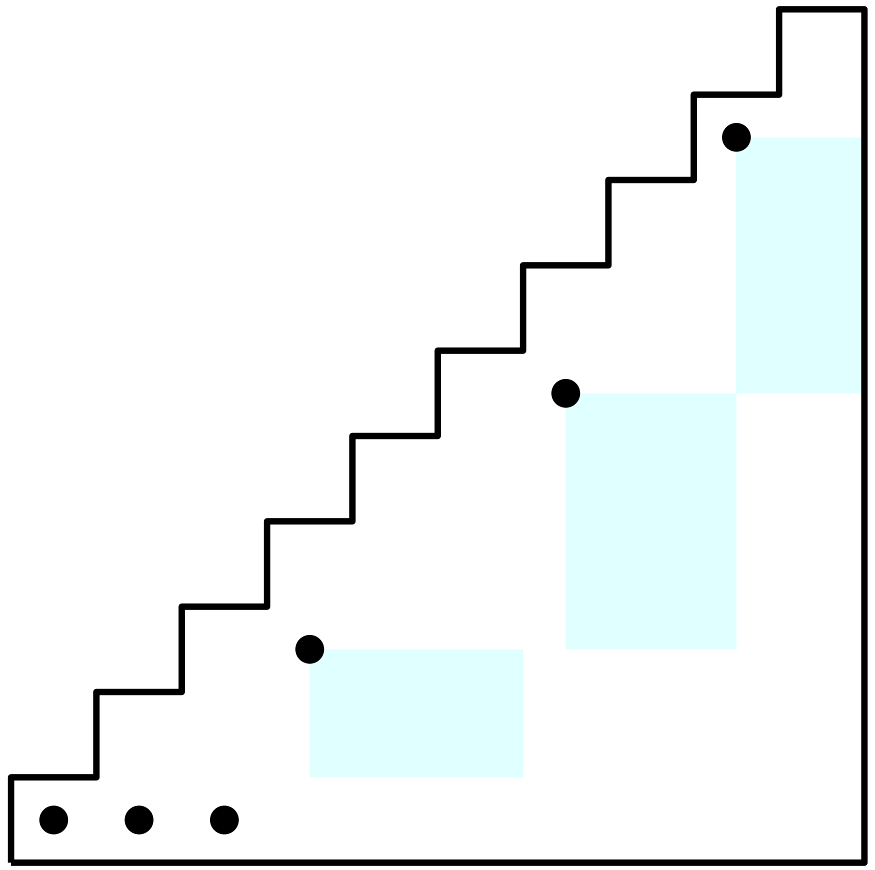
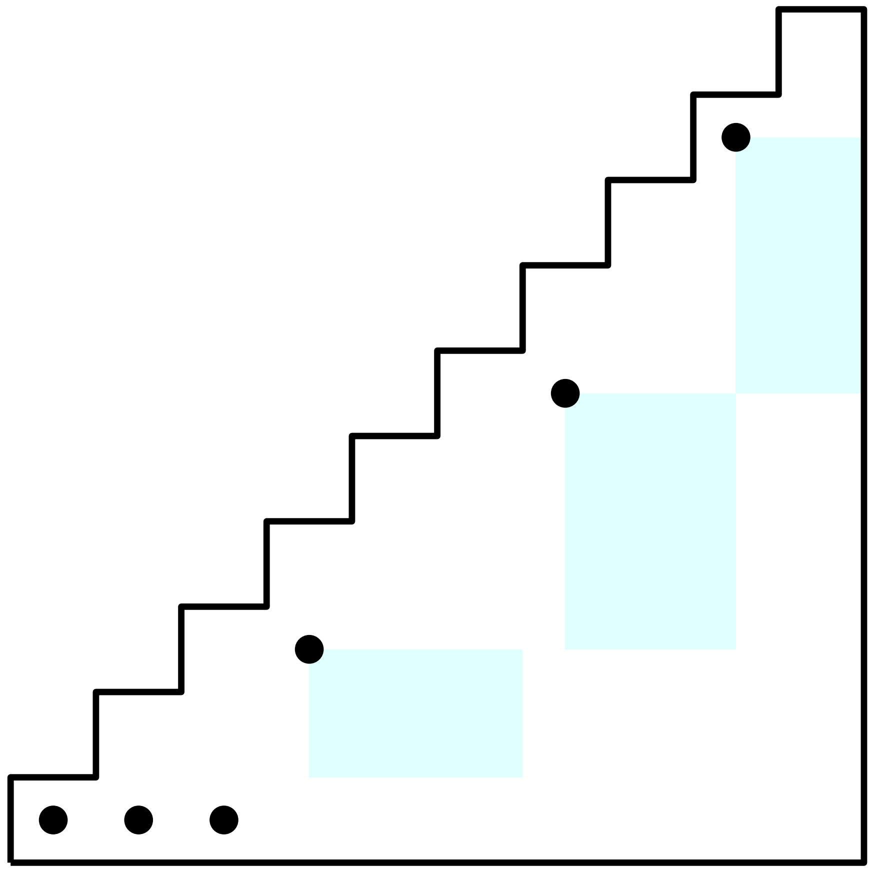


$I_n(010, 110, 120, 210)$

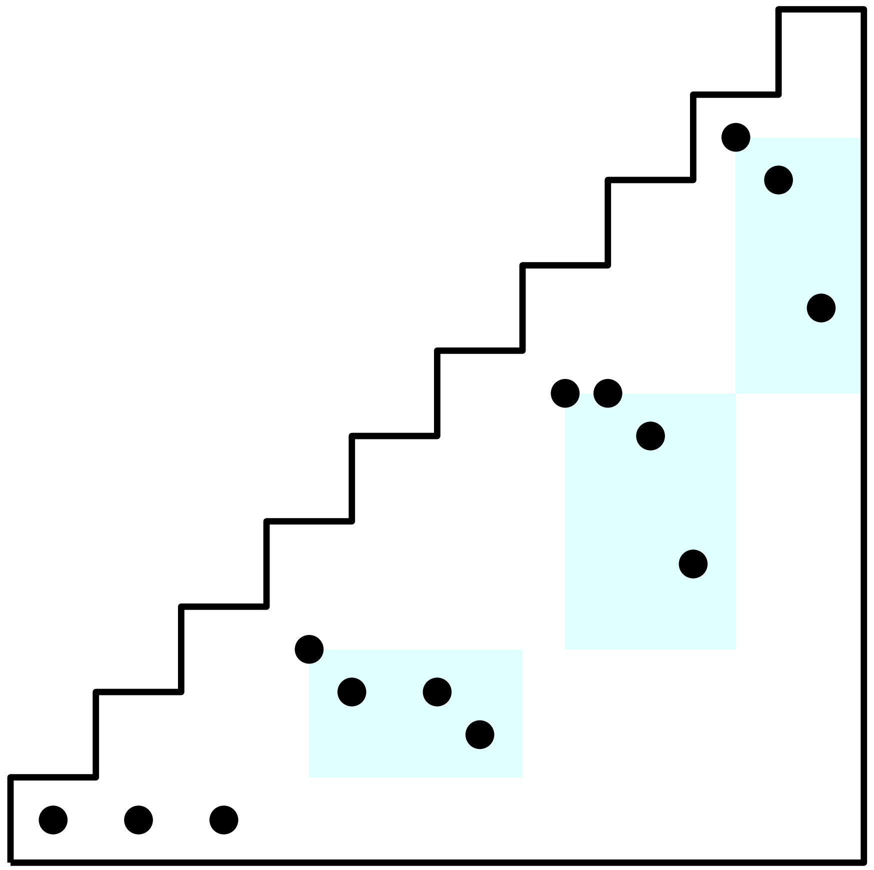


$I_n(010, 100, 120, 210)$

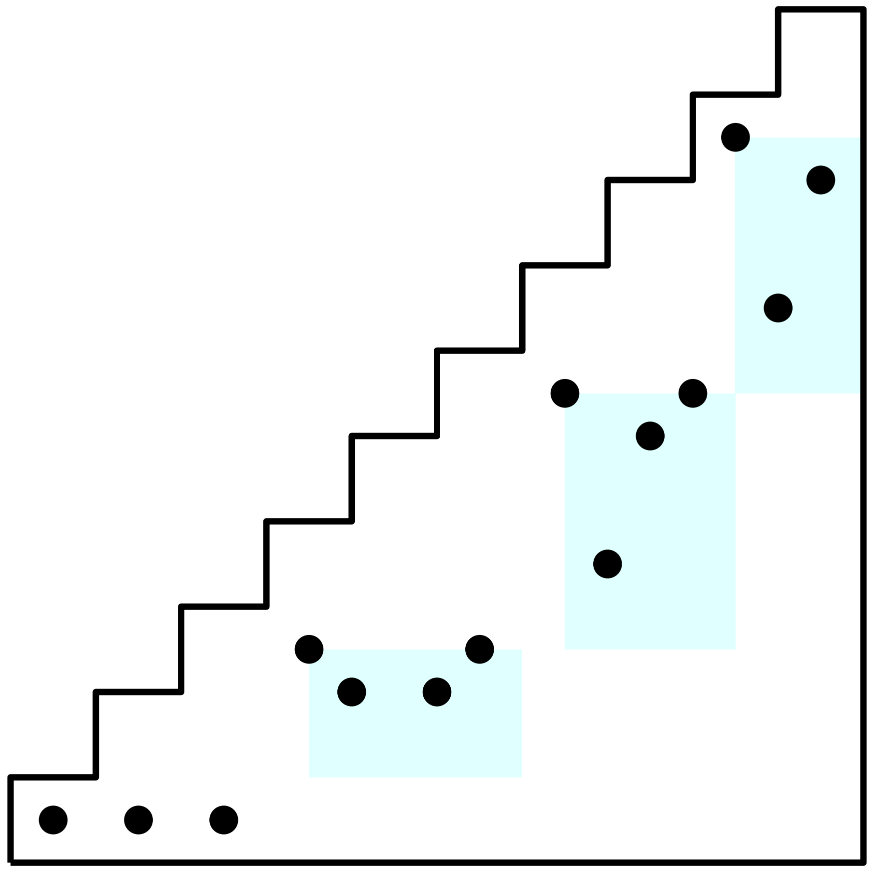


$|R^s_n(\top)| = |I_n(010, 101, 120, 201)|$, OEIS A279555 (Asinowski and P)

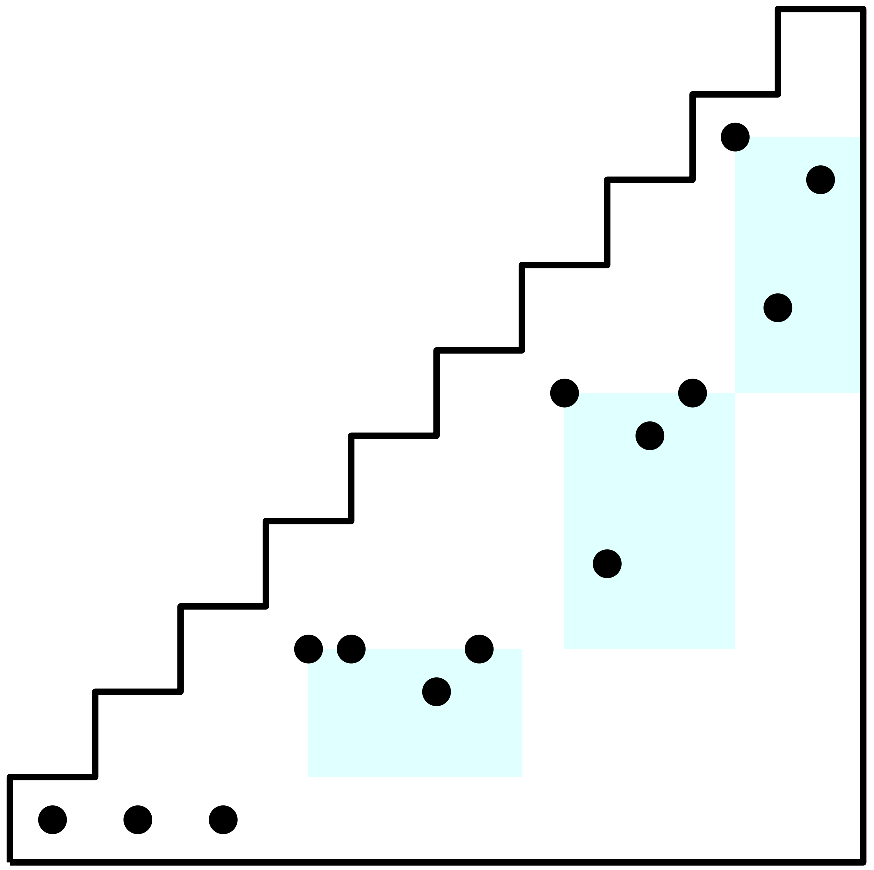
$I_n(010, 101, 120, 201)$



$I_n(010, 110, 120, 210)$



$I_n(010, 100, 120, 210)$



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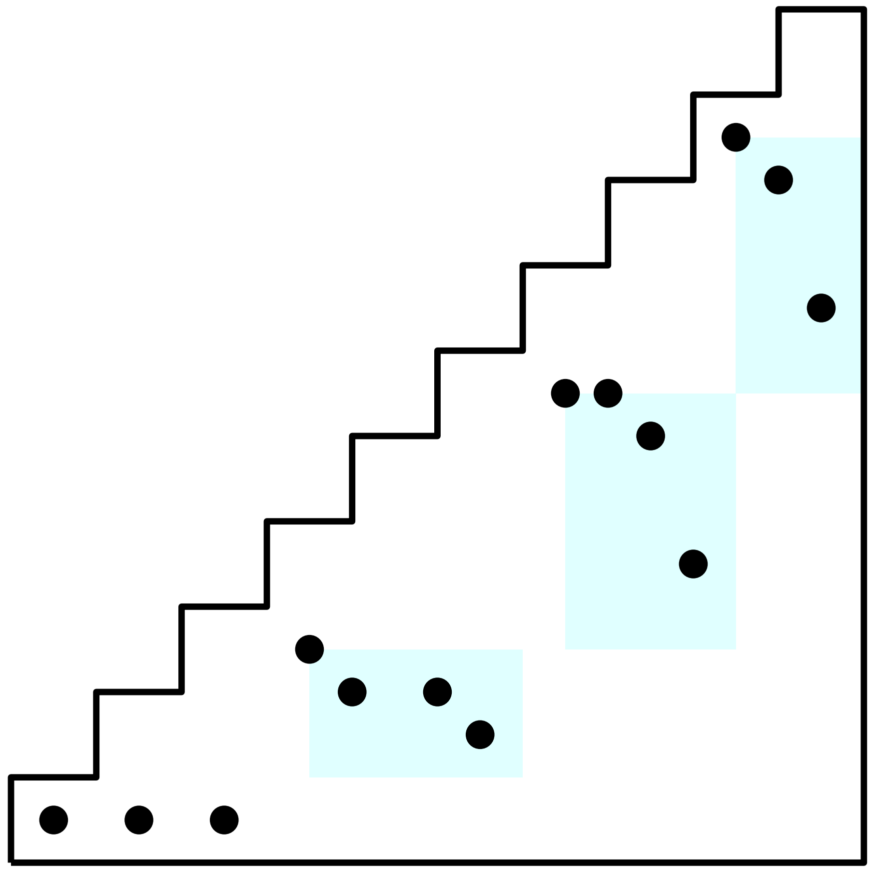
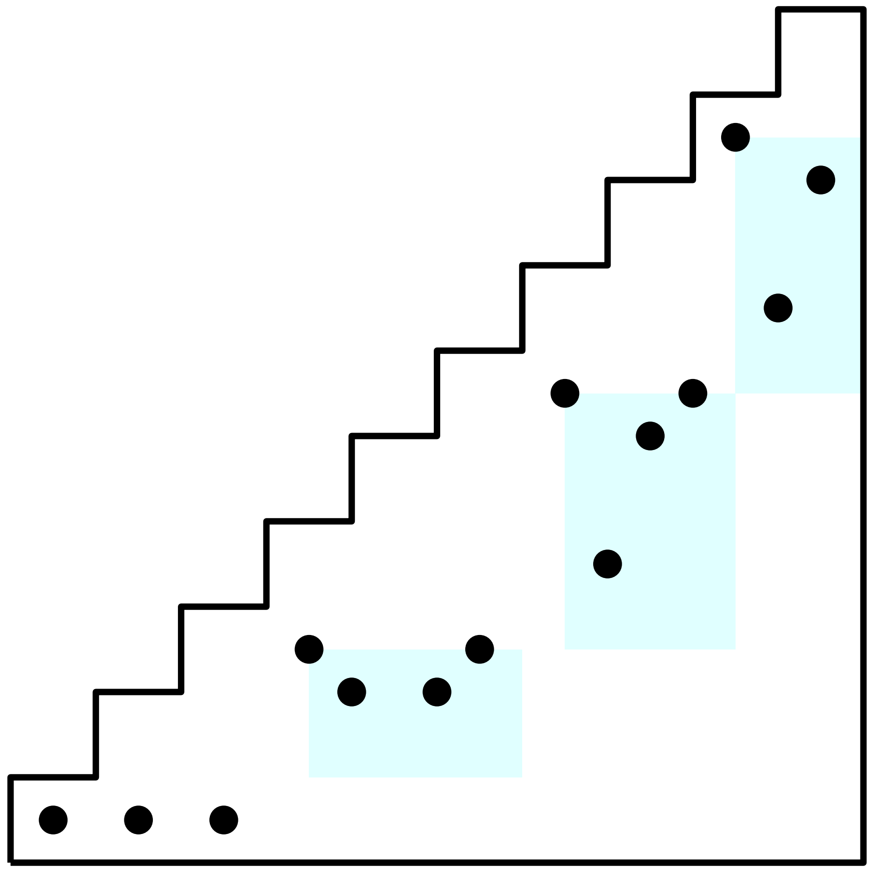
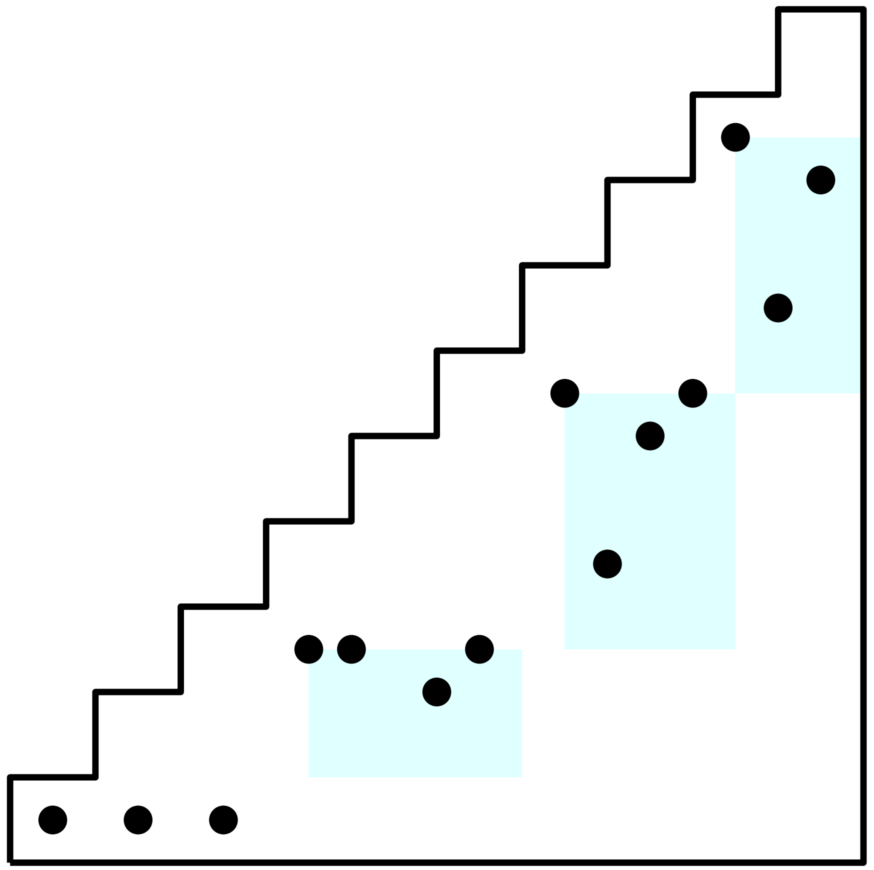
$I_n(010, 101, 120, 201)$



$I_n(010, 110, 120, 210)$

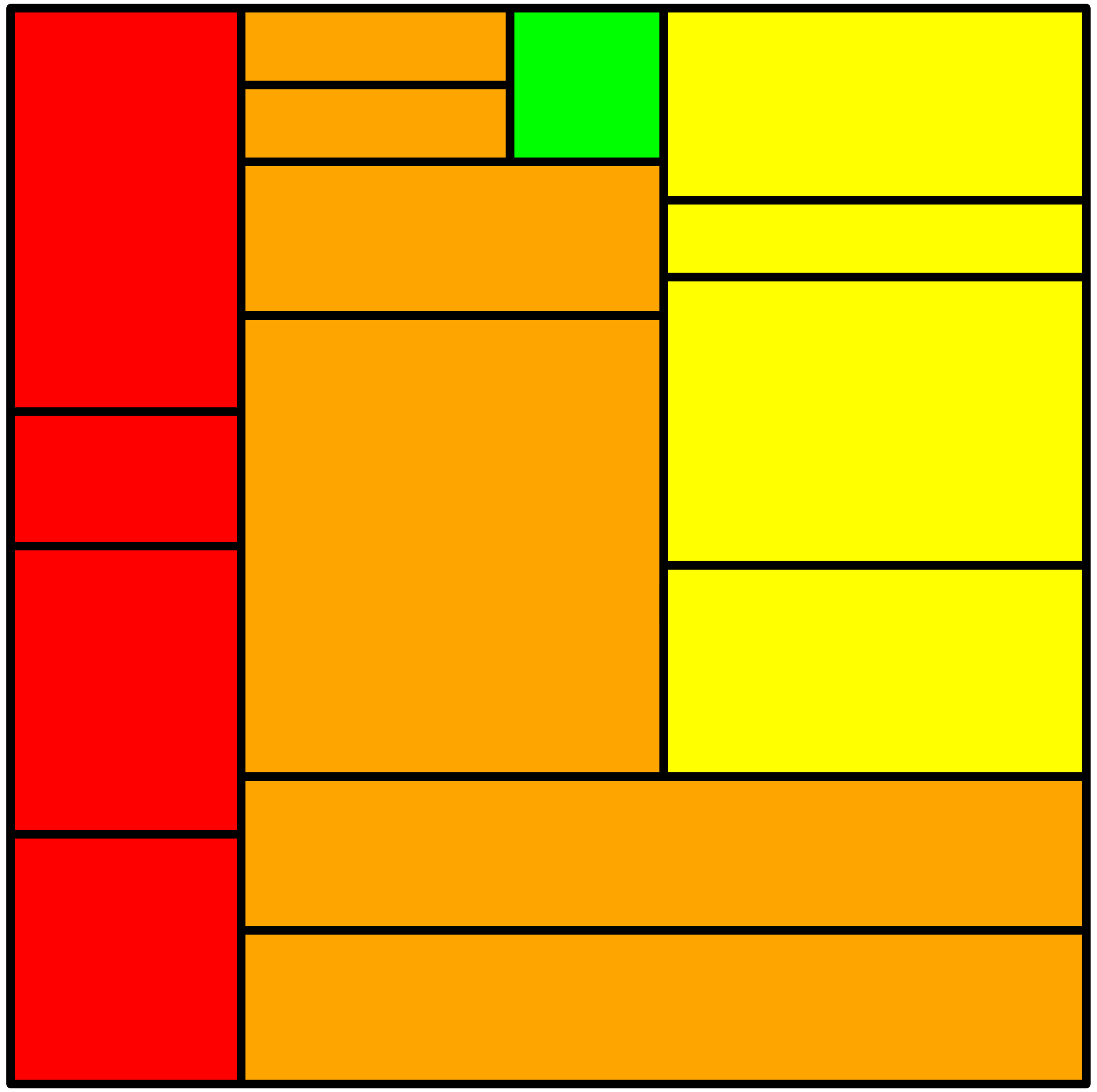


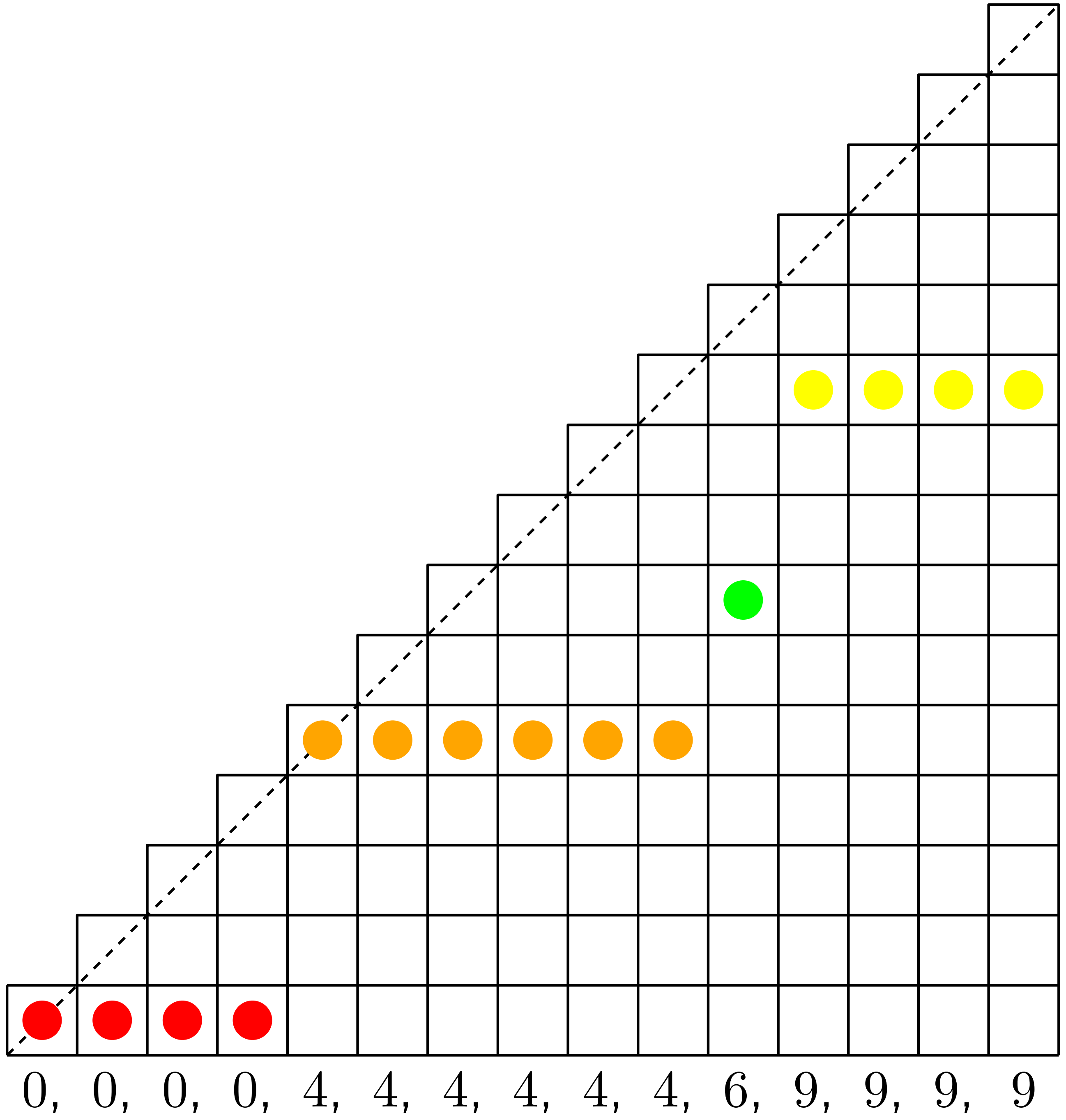
$I_n(010, 100, 120, 210)$



$|R_n^s(\top)| = |I_n(010, 101, 120, 201)|$, OEIS A279555 (Asinowski and P)

Proof: Bijection to inversion sequences





$|R^s_n(\top)| = |I_n(010, 101, 120, 201)|$, OEIS A279555 (Asinowski and P)

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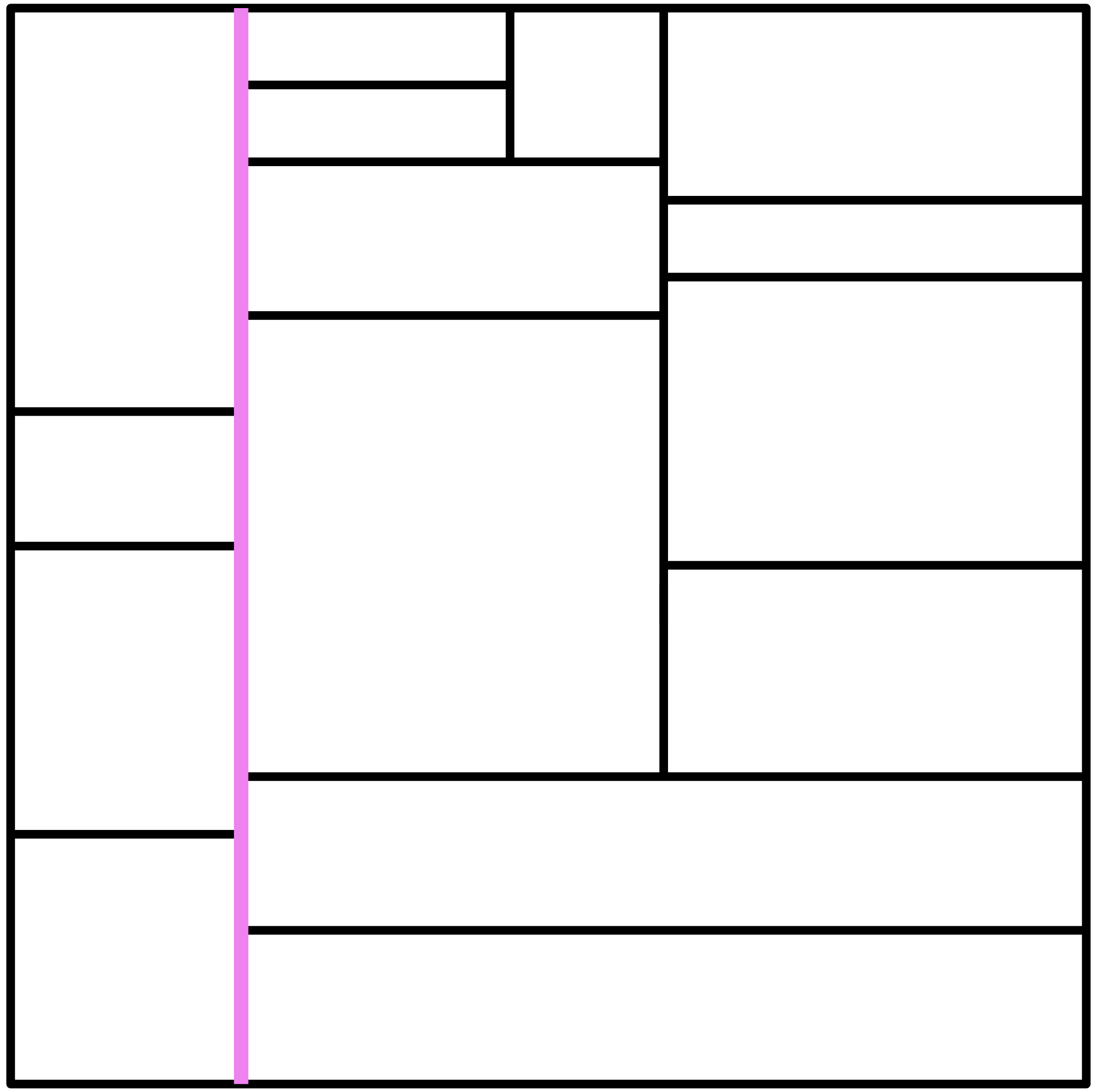

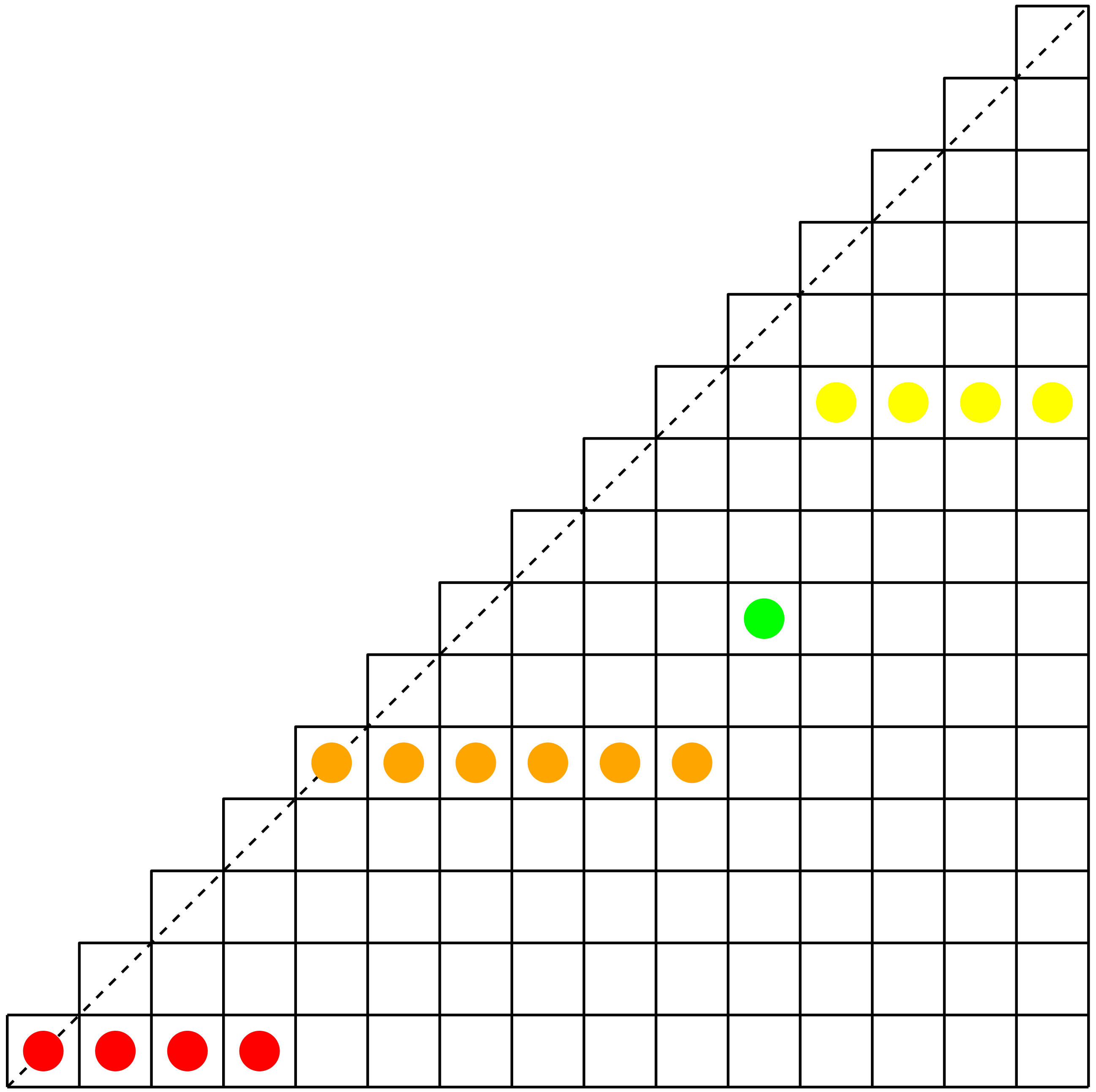

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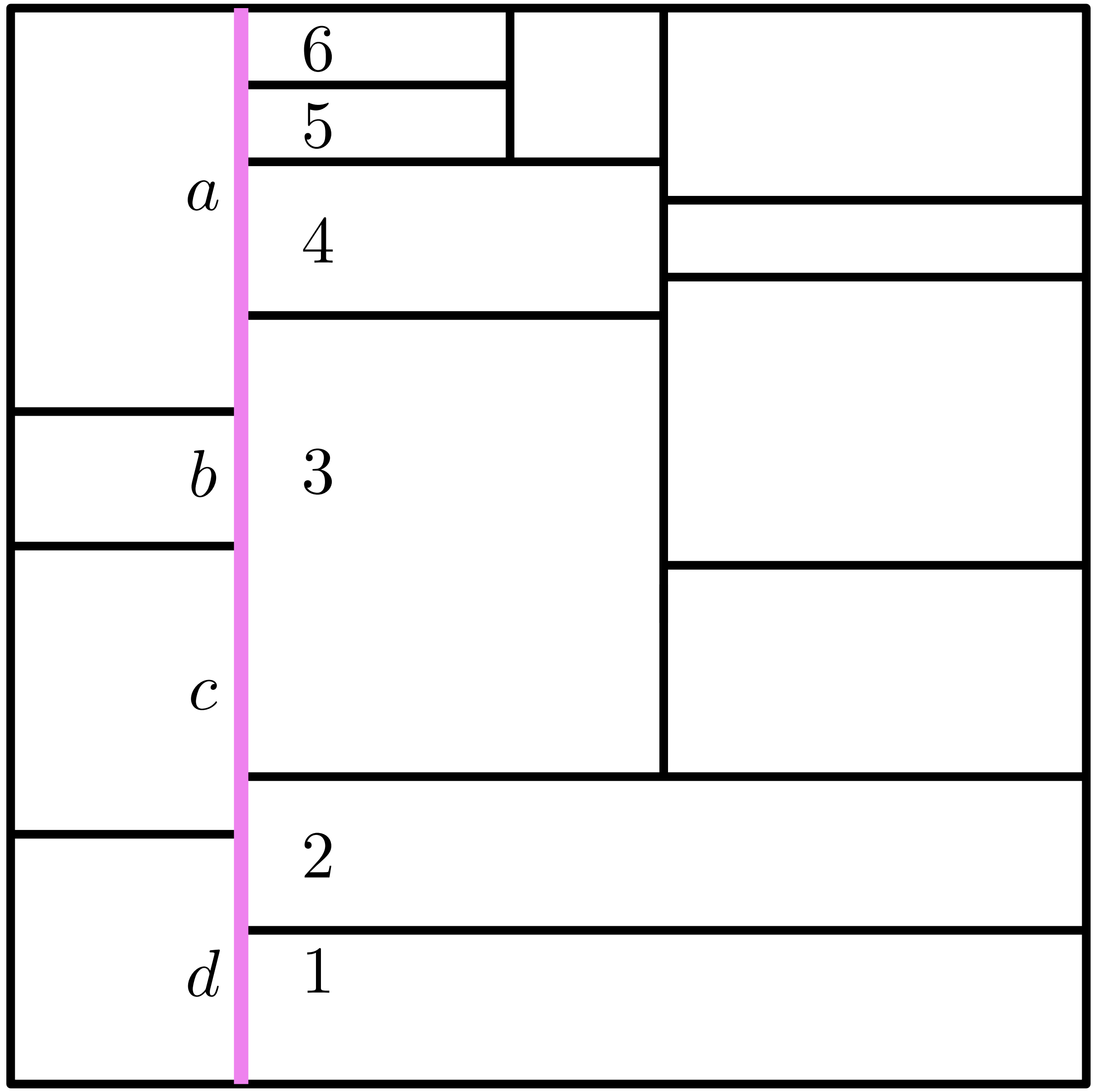

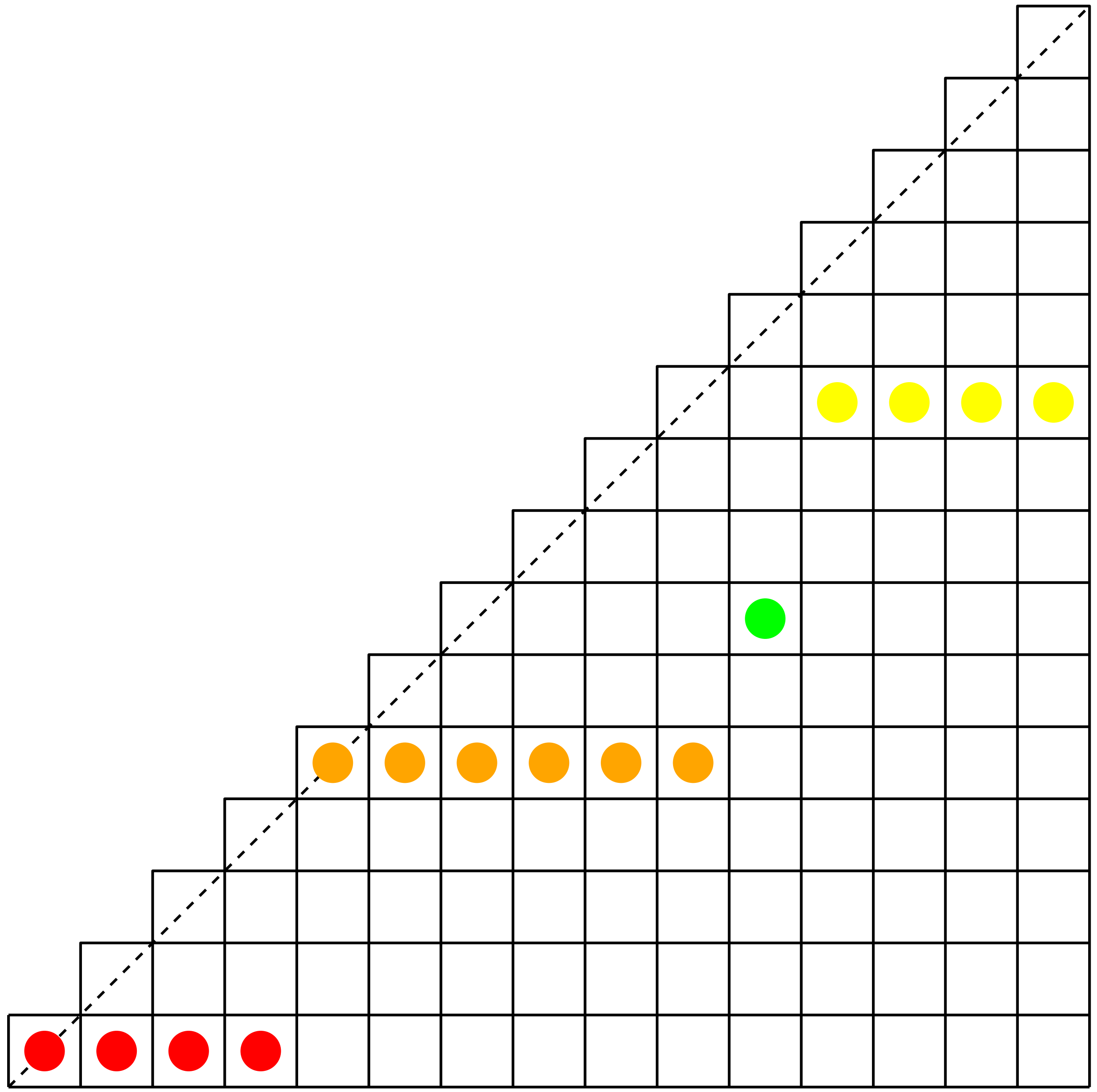

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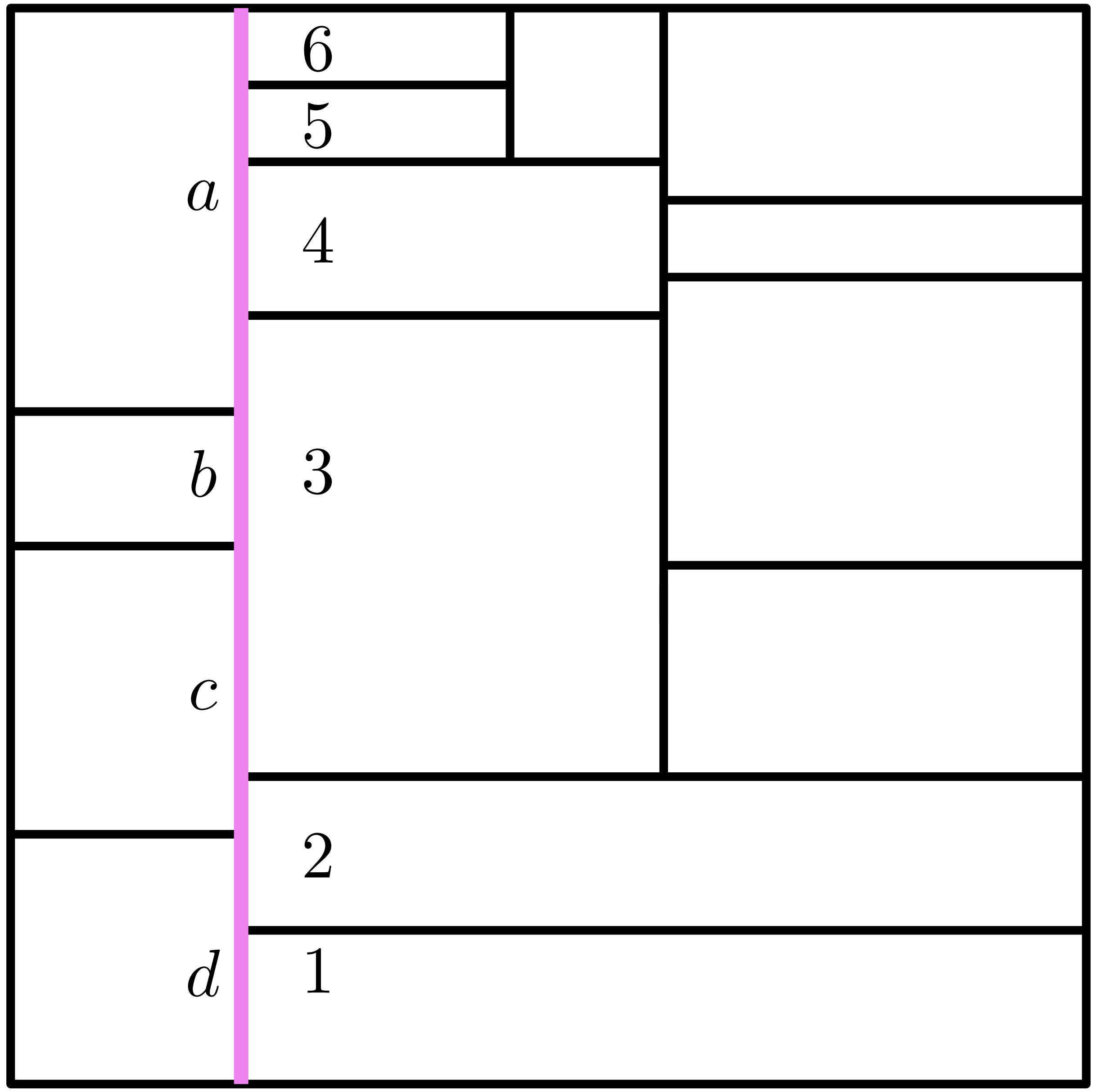

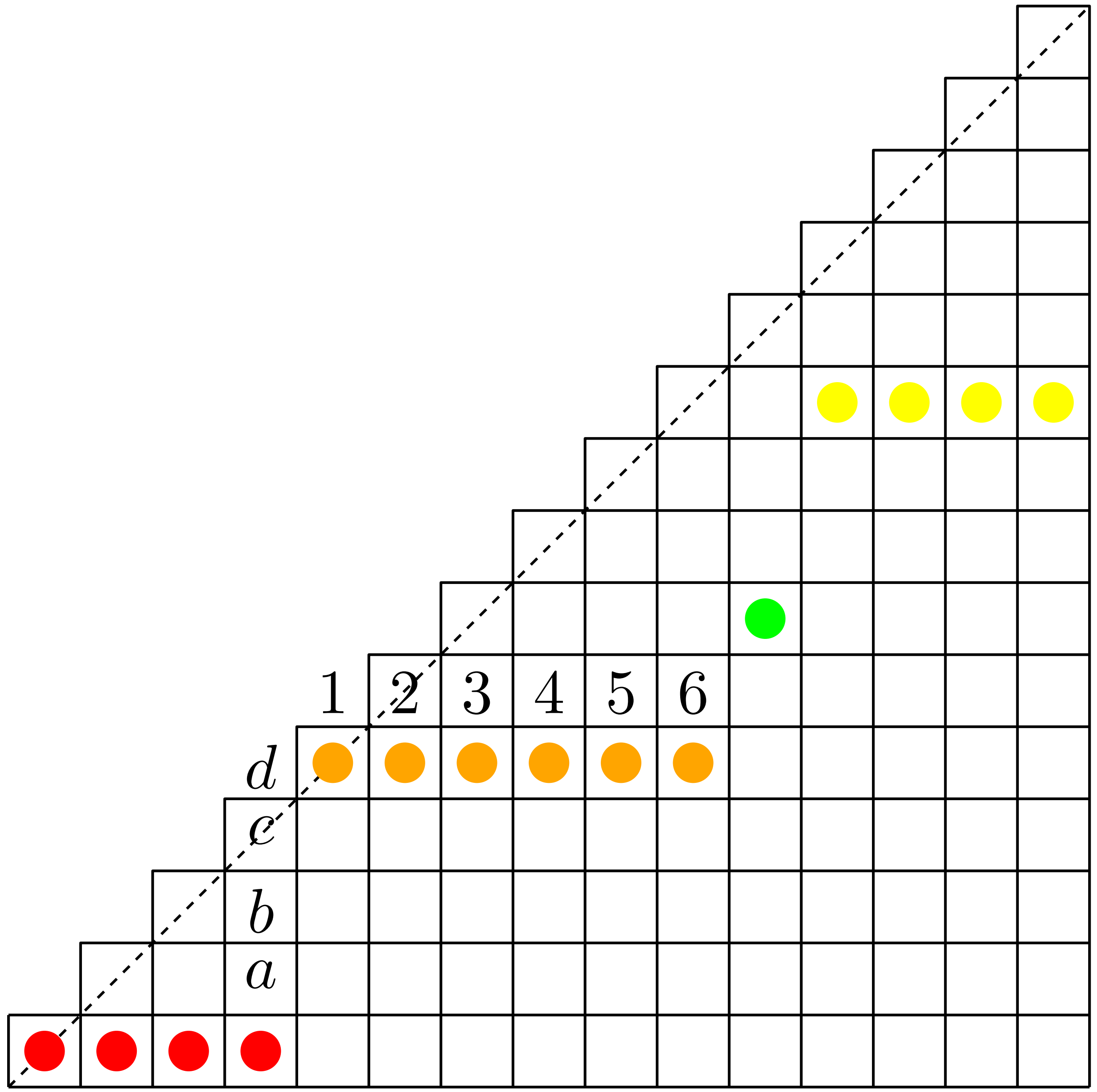

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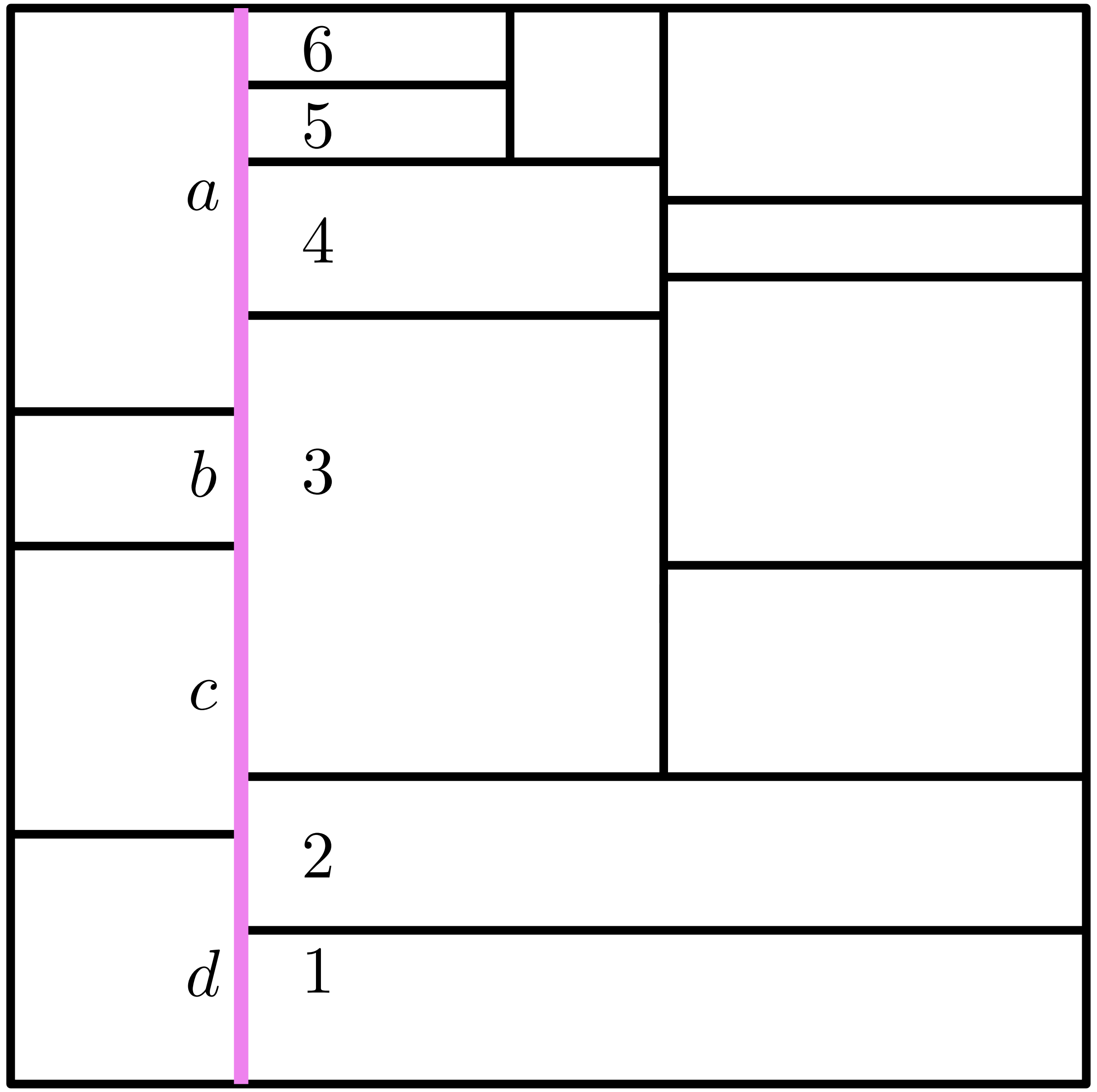

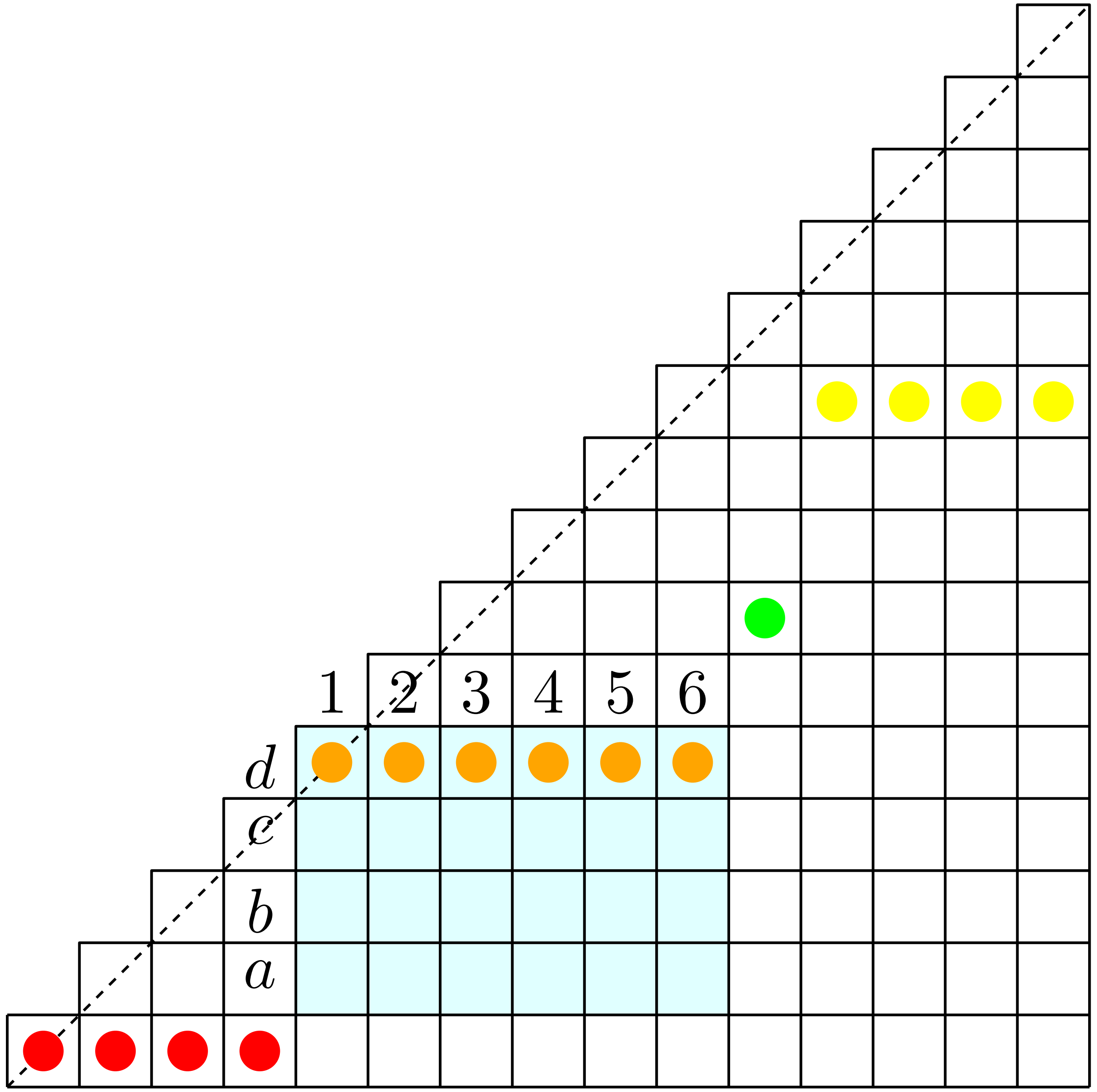

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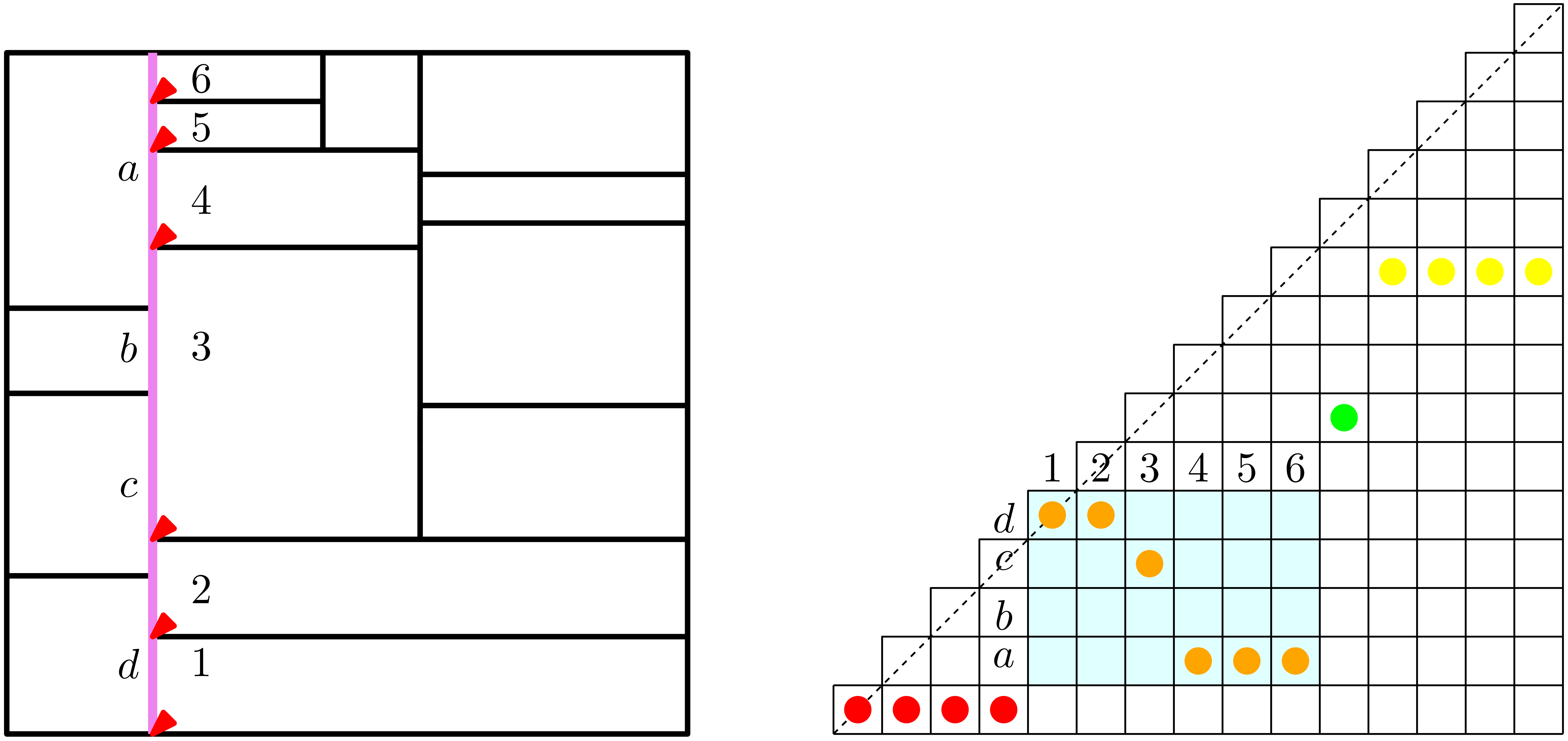

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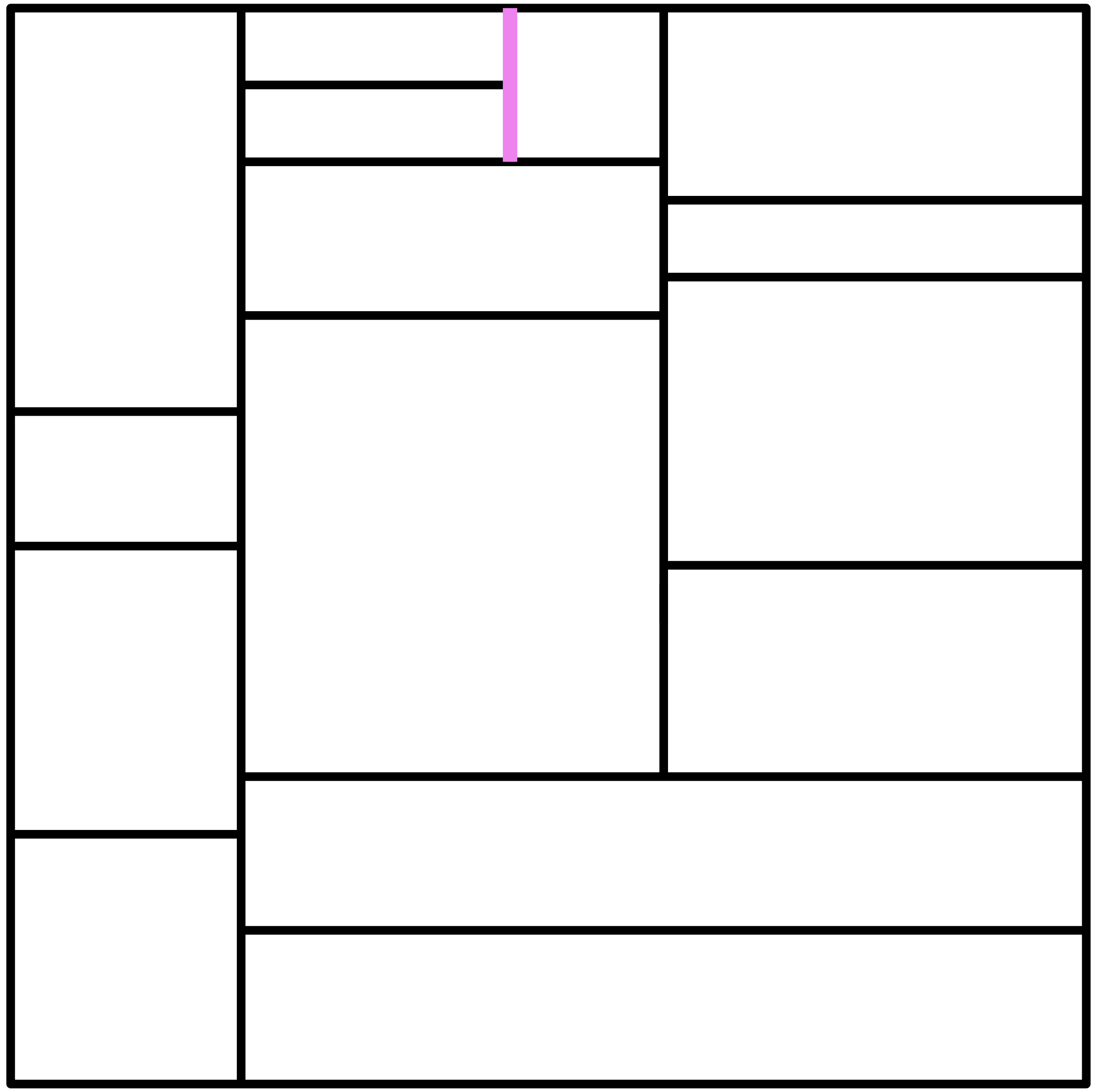

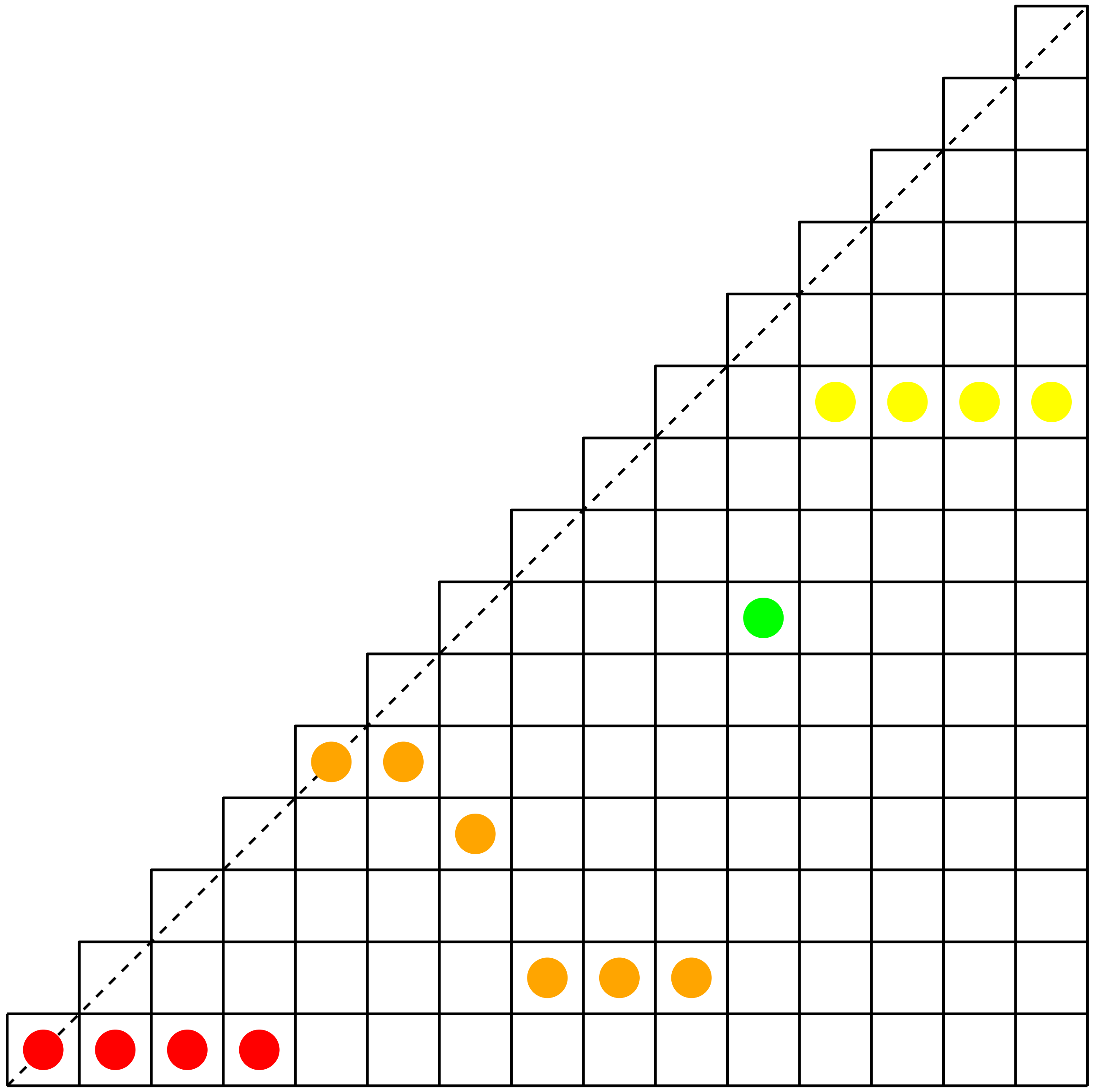

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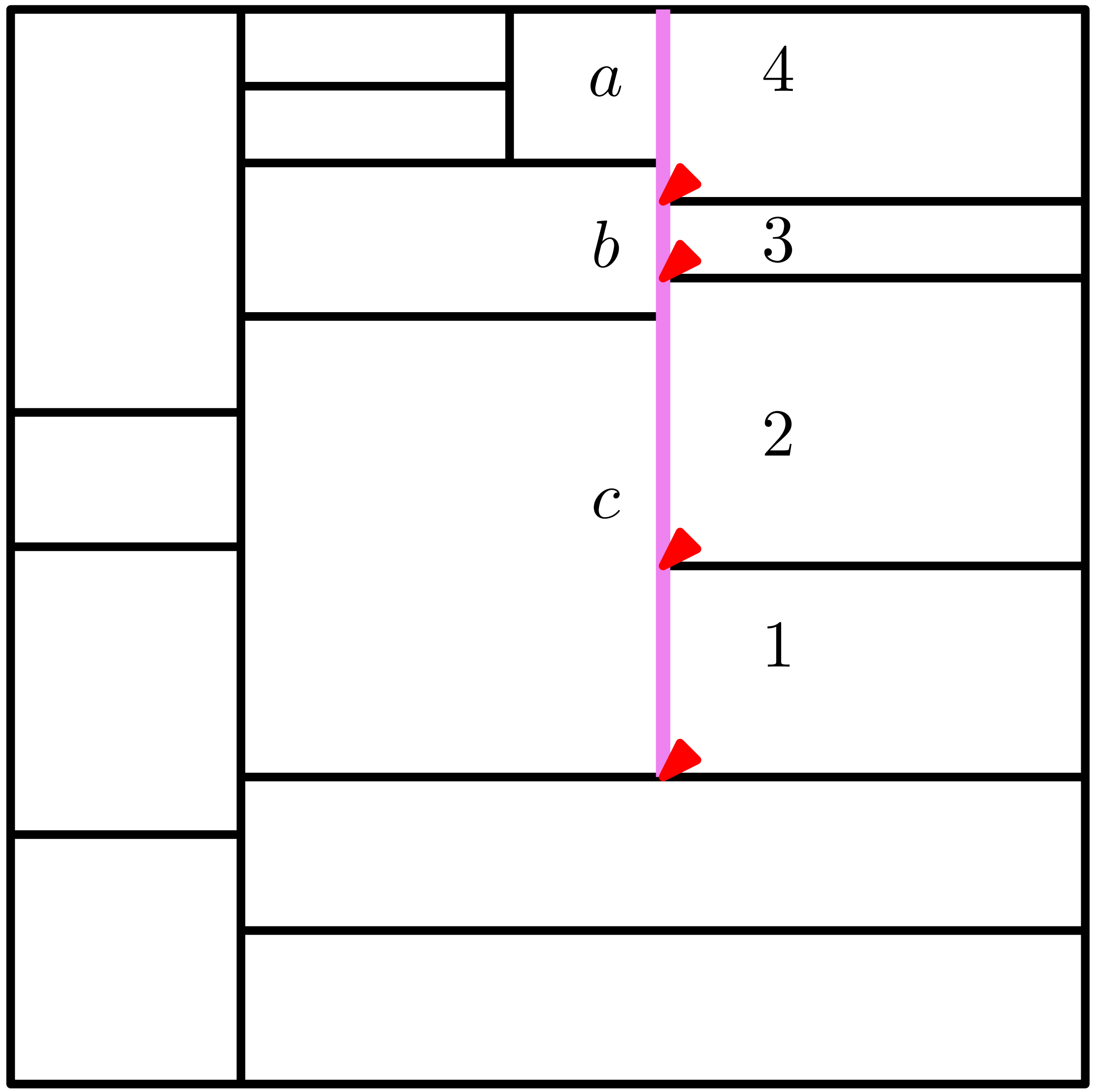

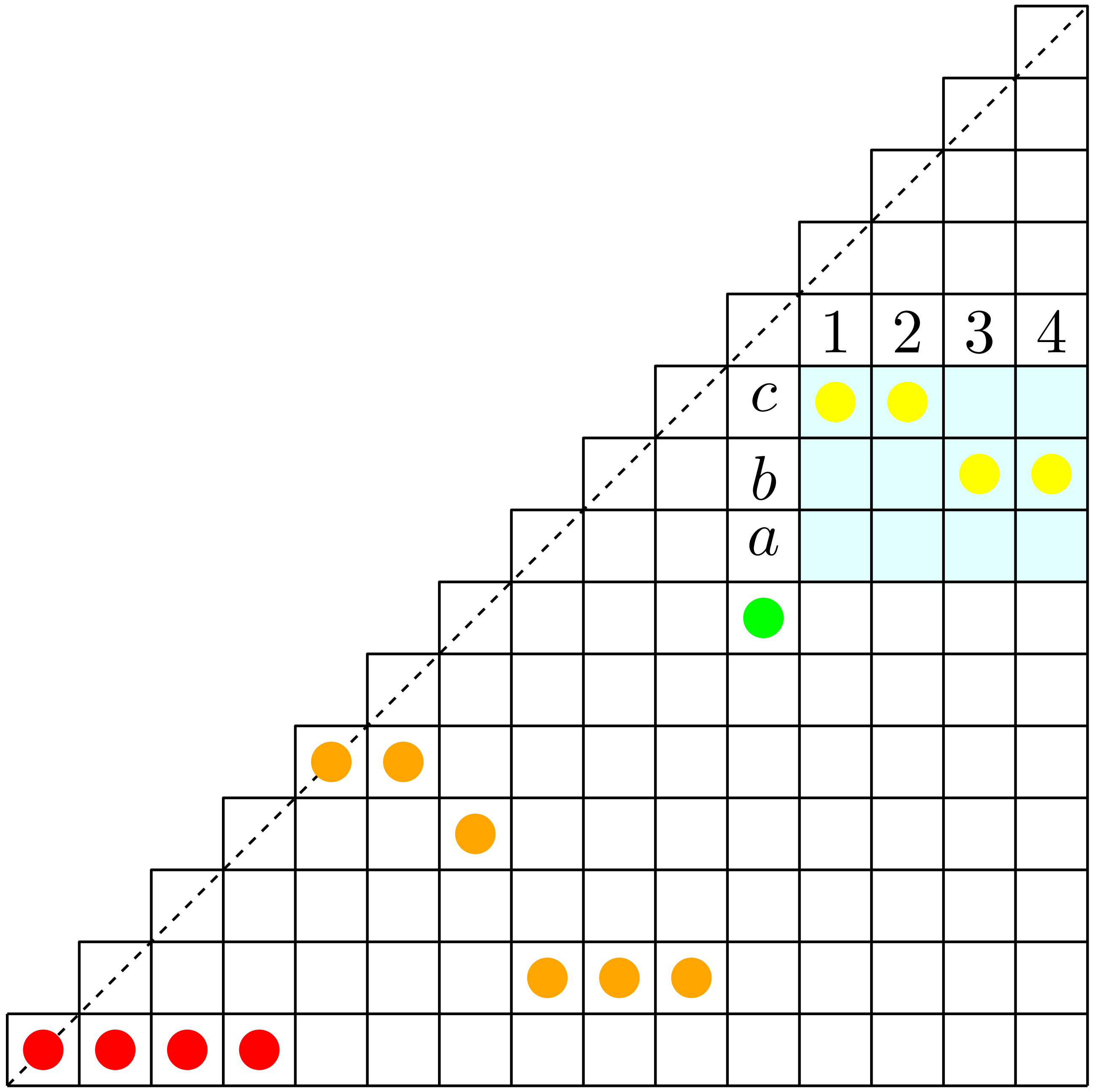

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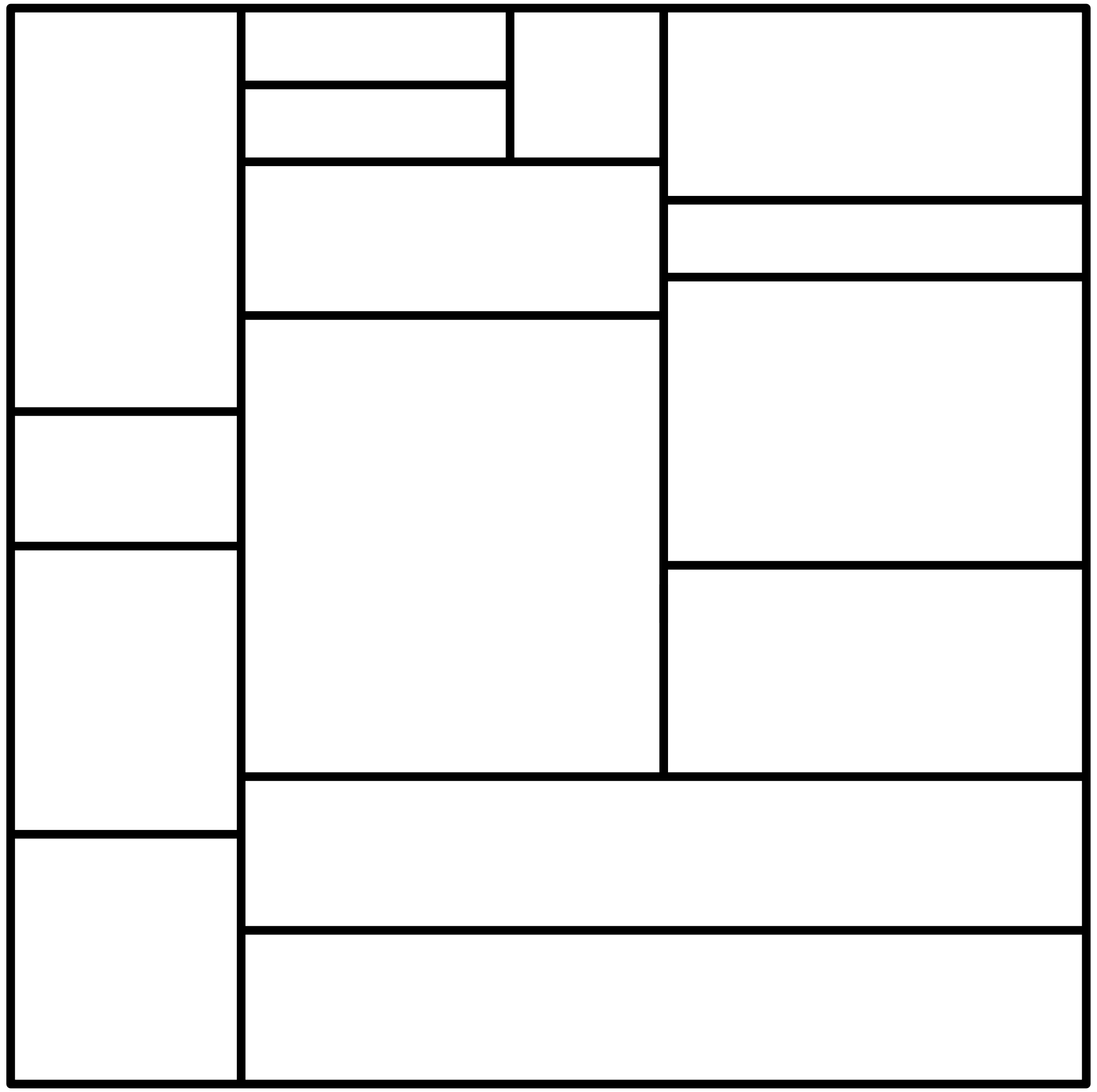
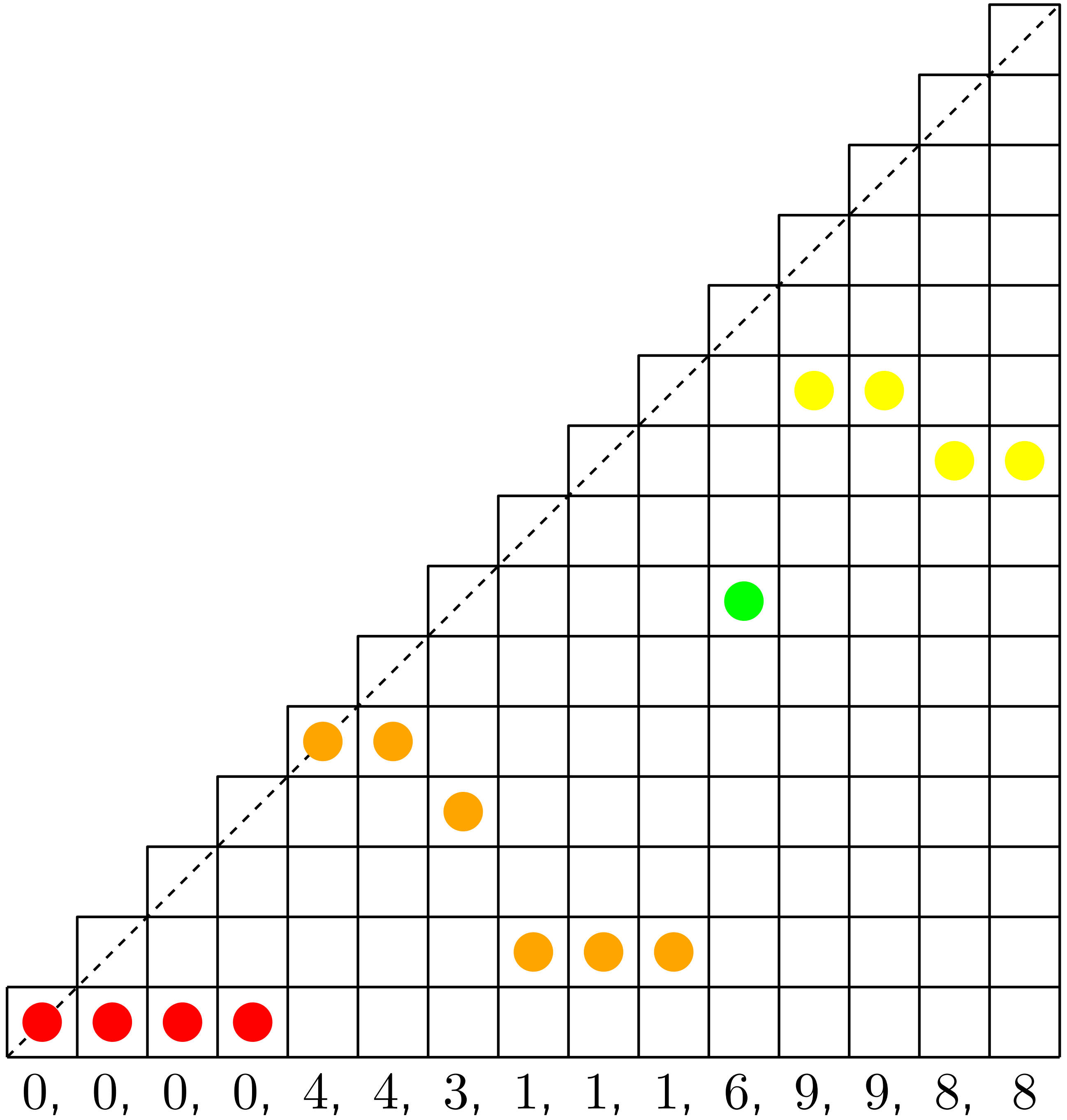




First geometric interpretation of sequence, sequence previously appeared in paper examining pattern avoidance in inversion sequences from Megan Martinez and Carla Savage (2018).

$I(010, 101, 120, 201)$, $I(011, 201)$, and $\top-$avoiding rectangulations

Theorem (Martinez & Savage 2018, Callan & Mansour 2023, Asinowski & P 2025)

$I(010, 101, 120, 201)$, $I(010, 100, 120, 210)$, $I(010, 110, 120, 210)$, and $\top-$avoiding rectangulations are all enumerated by A279555.

$I(010, 101, 120, 201)$, $I(011, 201)$, and $\top-$avoiding rectangulations

Theorem (Martinez & Savage 2018, Callan & Mansour 2023, Asinowski & P 2025)

$I(010, 101, 120, 201)$, $I(010, 100, 120, 210)$, $I(010, 110, 120, 210)$, and $\top-$avoiding rectangulations are all enumerated by A279555.

Conjecture (Yan & Lin 2020, Callan & Mansour 2023, Pantone 2024)

$I(011, 201)$ and $I(011, 210)$ are also enumerated by A279555.

Generating trees for $I(010, 101, 120, 201)$ and $I(011, 201)$ (Pantone, 2024)

Generating tree for $I(010, 101, 120, 201)$ (T1):

$$
\begin{aligned}
\text{Root :}\quad & (1,0). \\
\text{Succession rules :}\quad & (k,\ell) \longrightarrow (1,k-1),\ (2,k-2),\ \ldots,\ (k,0); && (*) \\
& \qquad\qquad (k+1,\ell),\ (k+1,\ell-1),\ \ldots,\ (k+1,0). && (**)
\end{aligned}
$$

Generating tree for $I(011, 201)$ (T2):

$$
\begin{aligned}
\text{Root :}\quad & (1,0). \\
\text{Succession rules :}\quad & (k,\ell) \longrightarrow (1,k+\ell-1),\ (2,k+\ell-2),\ \ldots,\ (k,\ell); && (*) \\
& \qquad\qquad (k+1,\ell-1),\ (k+1,\ell-2),\ \ldots,\ (k+1,0); && (**) \\
& \qquad\qquad (k+1,0). && (***)
\end{aligned}
$$

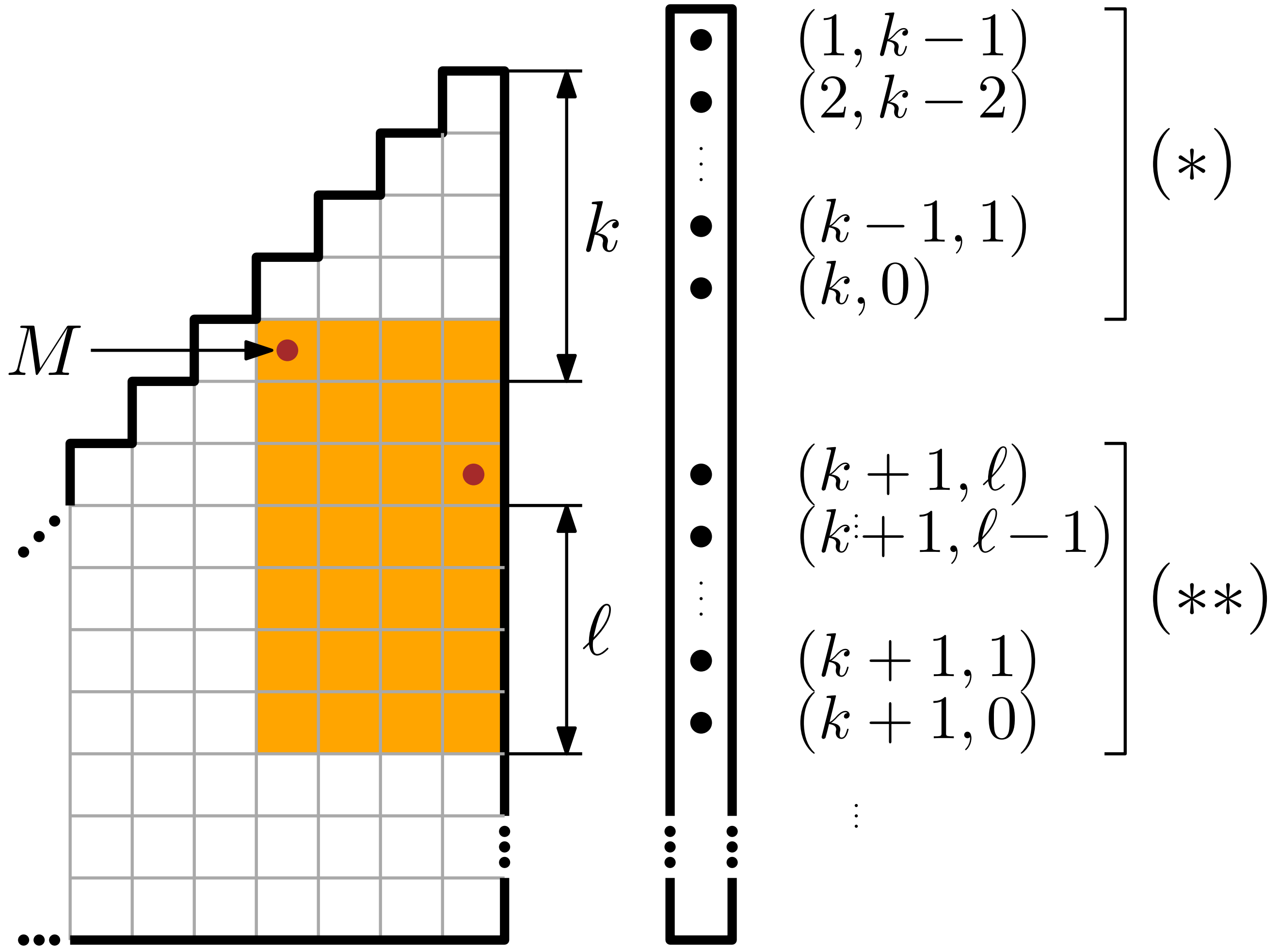
Here, k is the *bounce* defined as $n - M$, where n is the length and M is its maximal value;
ℓ in T1 is the number of admissible values j such that $0 < j < e_n$,
ℓ in T2 is the number of admissible values j such that $0 < j < M$.

T1: Generating tree for $I(010, 101, 120, 201)$ and $\top-$avoiding rectangulations

Root : $(1, 0)$.

Succession rules : $(k, \ell) \longrightarrow (1, k-1),\ (2, k-2),\ \ldots,\ (k, 0);$ $(*)$

$(k+1, \ell),\ (k+1, \ell-1),\ \ldots,\ (k+1, 0).$ $(**)$



$k = n - M$ bounce

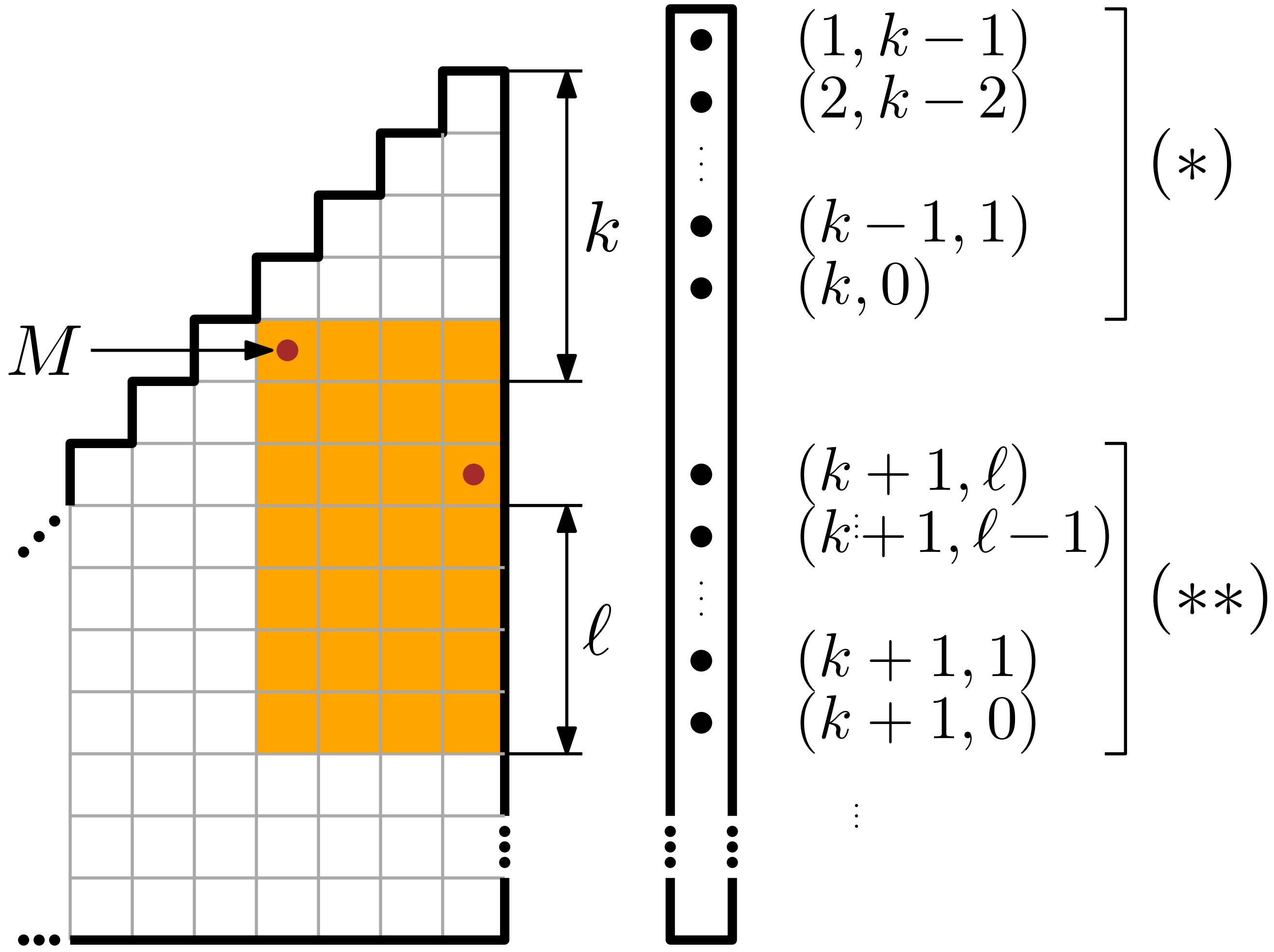
ℓ admissible values j, $0 < j < e_n$.

T1: Generating tree for $I(010, 101, 120, 201)$ and $\top$−avoiding rectangulations

Root : $(1, 0)$.

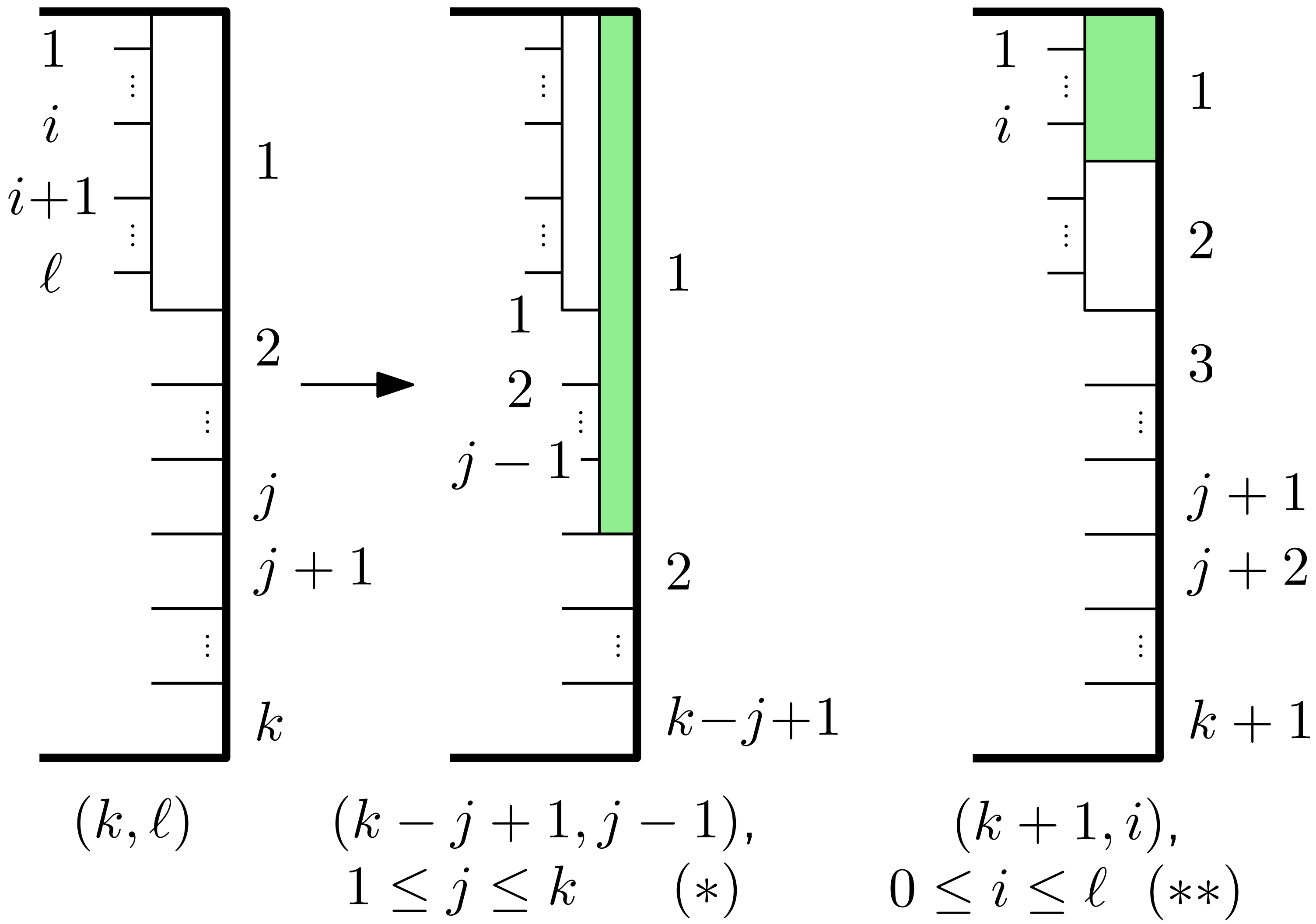
Succession rules : $(k, \ell) \longrightarrow (1, k-1),\ (2, k-2),\ \dots,\ (k, 0);$ $\quad (*)$

$(k+1, \ell),\ (k+1, \ell-1),\ \dots,\ (k+1, 0).$ $\quad (**)$



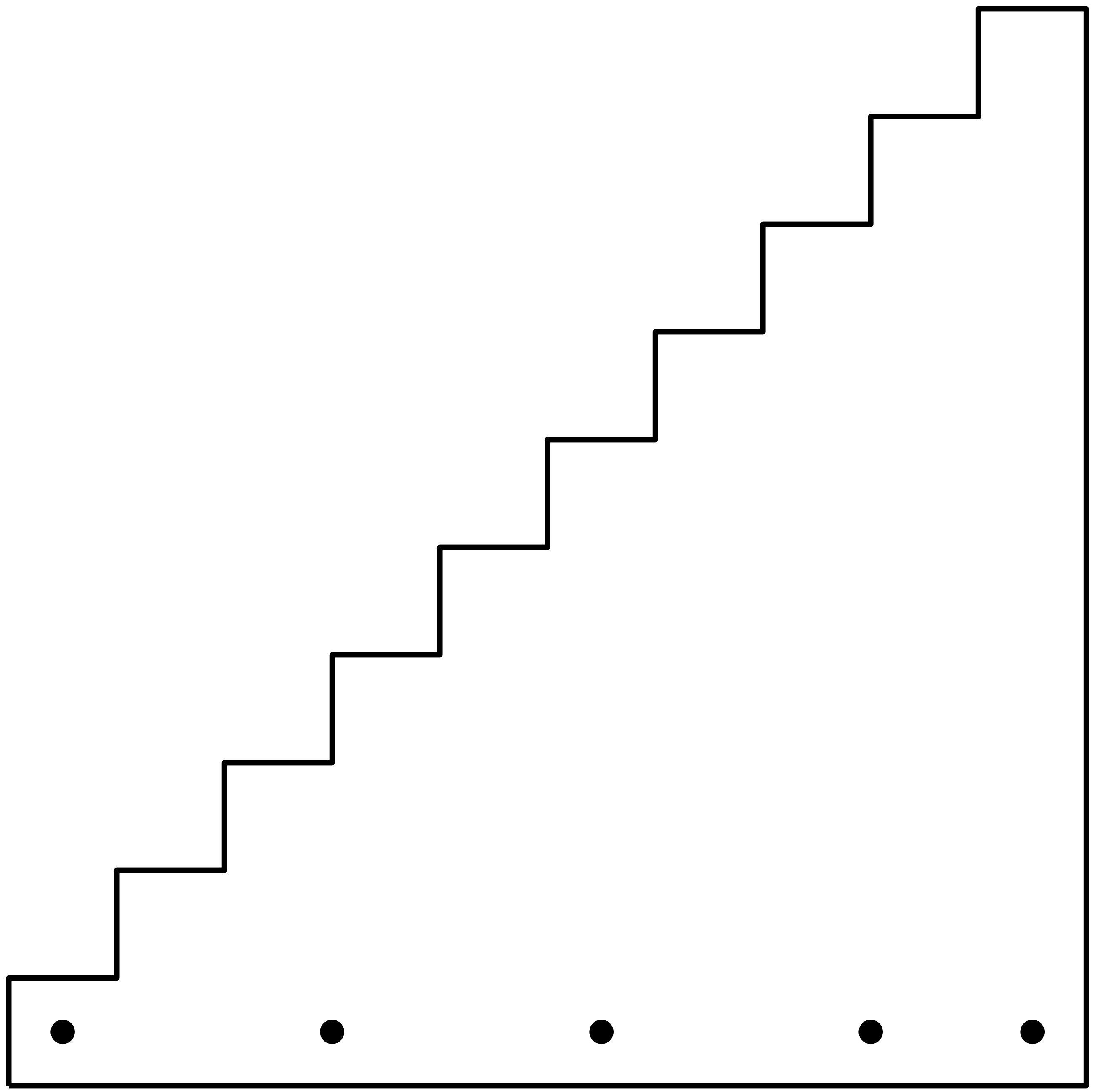
$k = n - M$ bounce

ℓ admissible values j, $0 < j < e_n$.



k rectangles touch E; ℓ segments touch NE on the left

T2: Generating tree for $I(011, 201)$ and $\perp$-avoiding rectangulations



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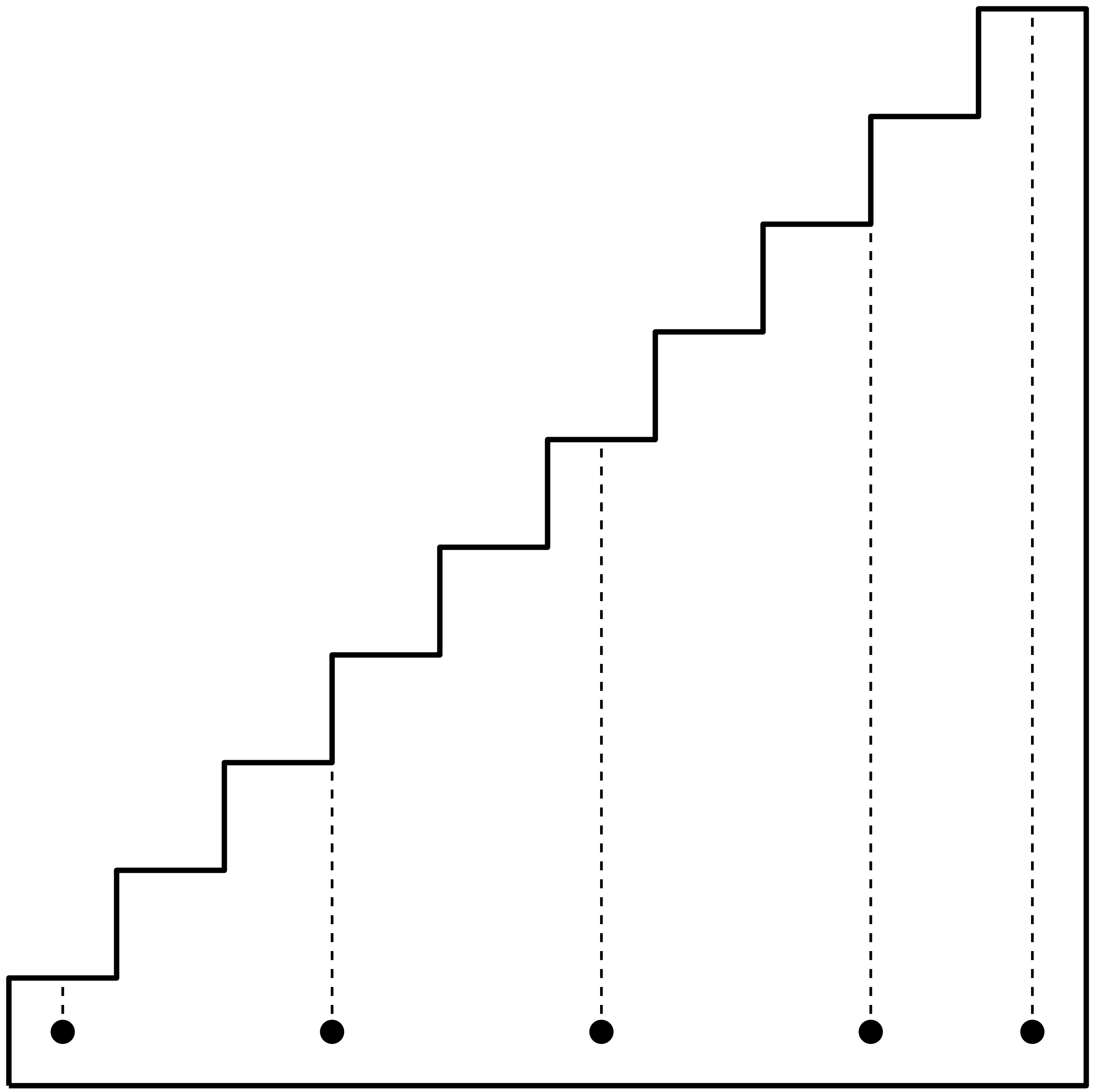

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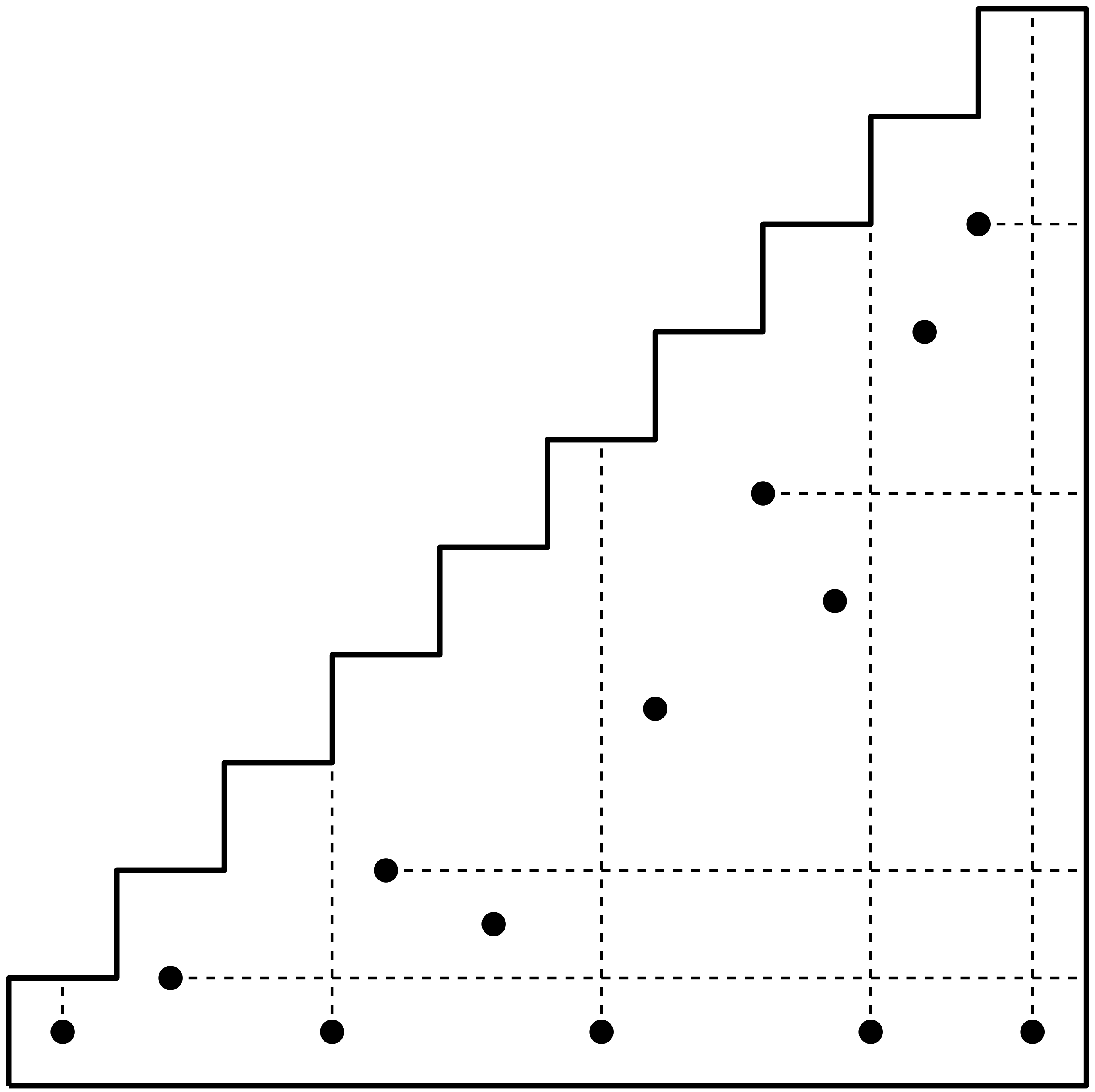
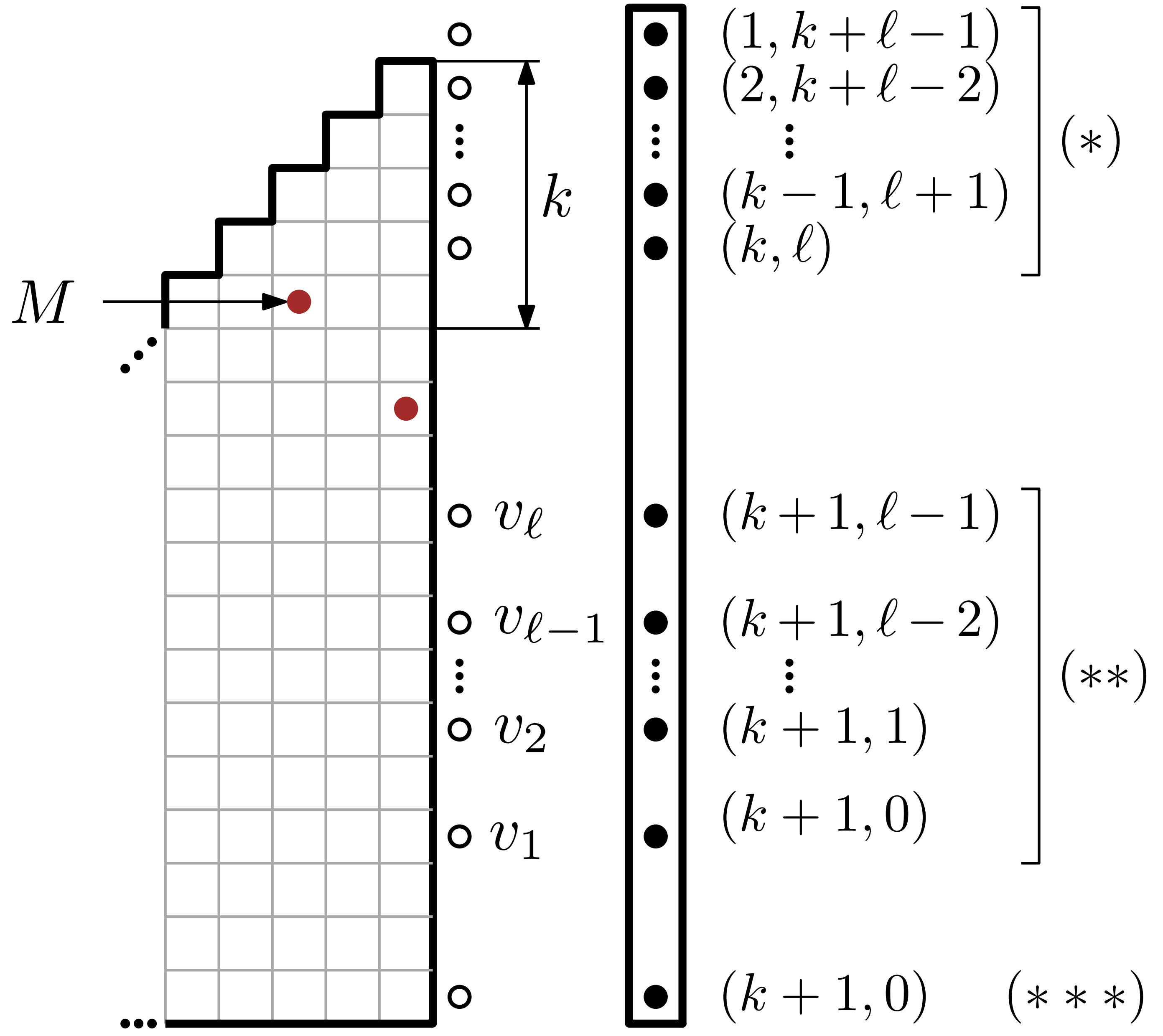


T2: Generating tree for $I(011, 201)$ and $\perp$-avoiding rectangulations

$$\begin{array}{rlll} \text{Root}: & (1,0). & & \\ \text{Succession rules}: & (k,\ell) & \longrightarrow & (1,k+\ell-1),\ (2,k+\ell-2),\ \ldots,\ (k,\ell); \quad (*) \\ & & & (k+1,\ell-1),\ (k+1,\ell-2),\ \ldots,\ (k+1,0); \quad (**) \\ & & & (k+1,0). \quad (***) \end{array}$$



T2: Generating tree for $I(011, 201)$ and $\perp$-avoiding rectangulations

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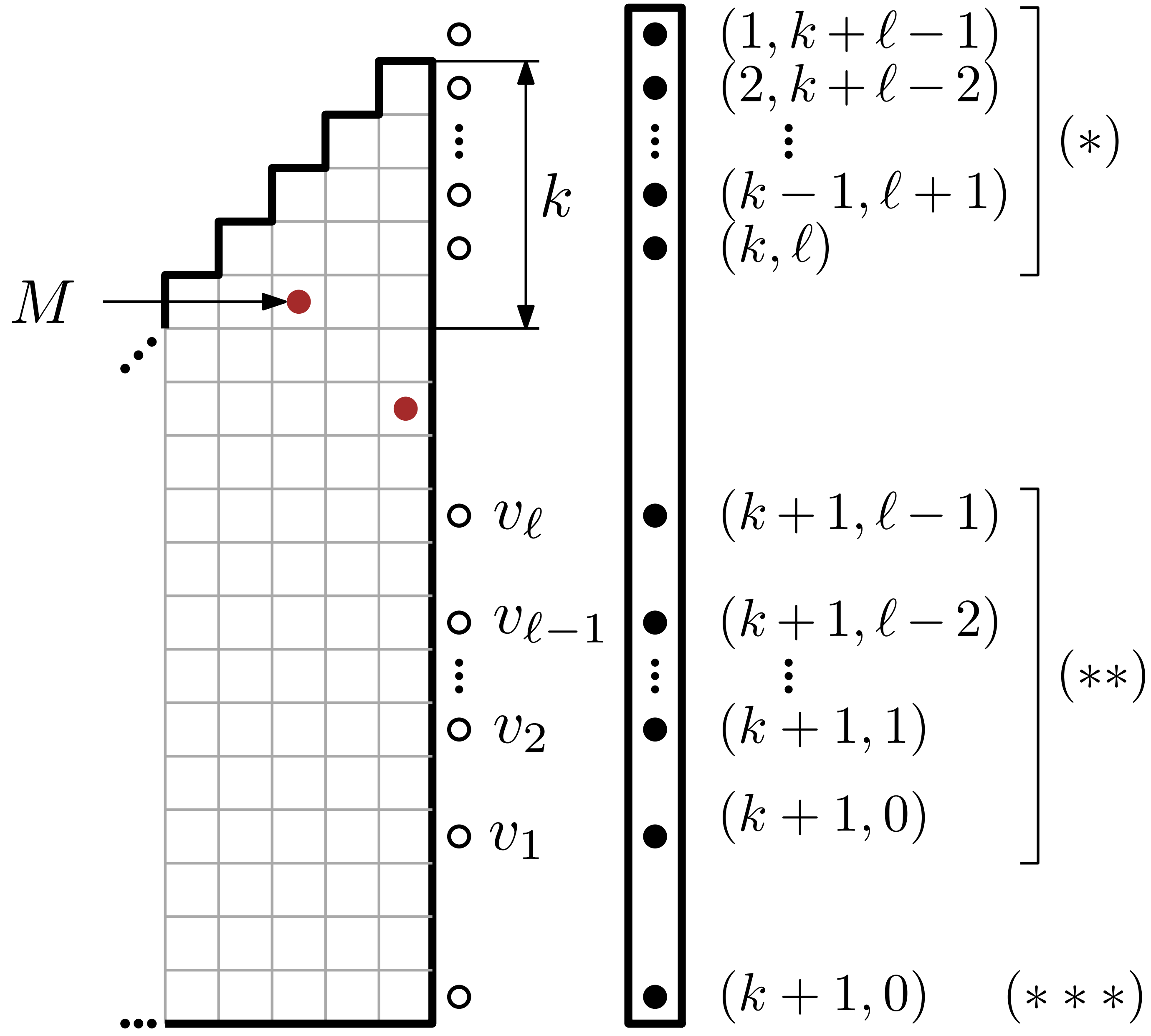
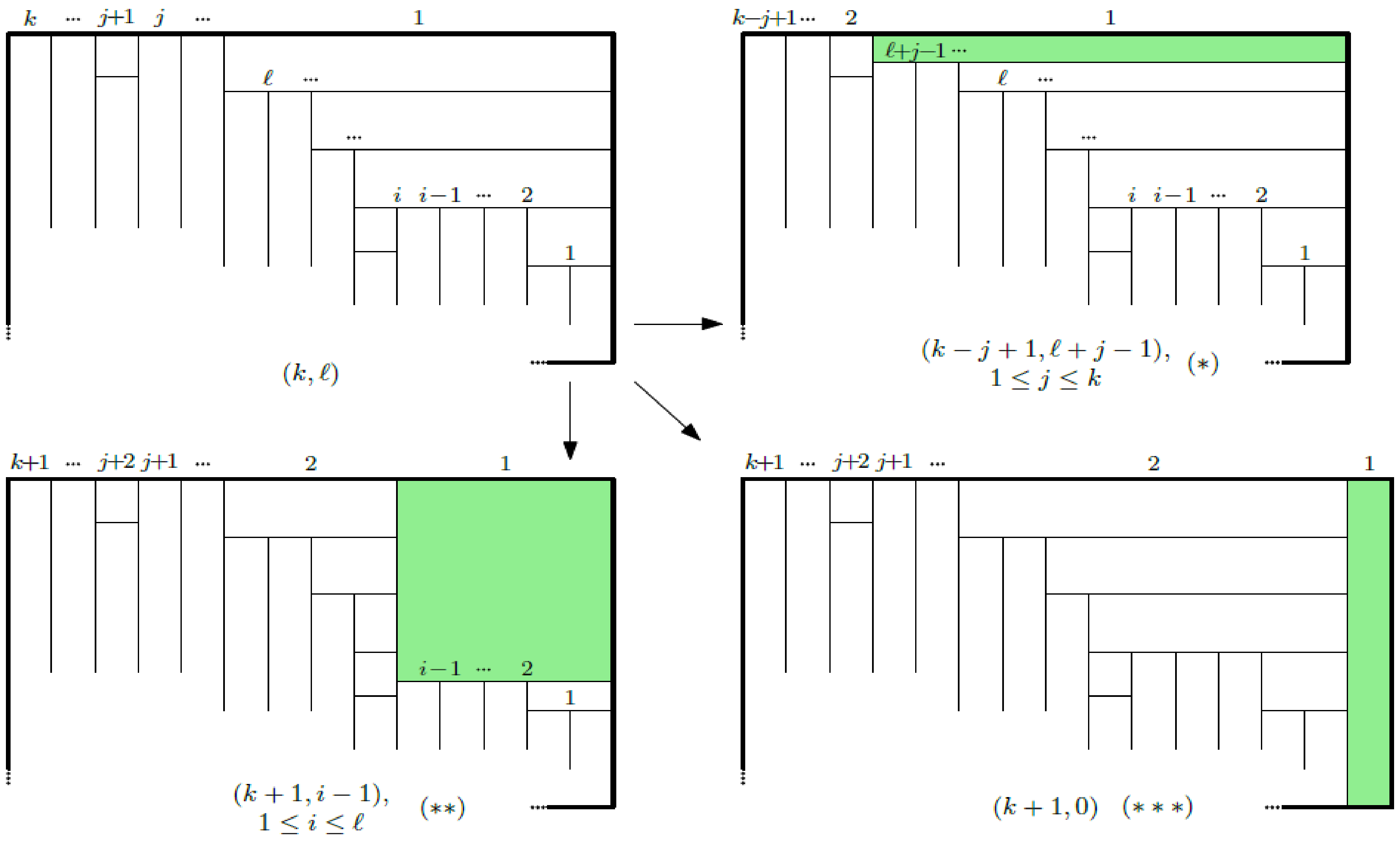
$(k+1, 0).$ $(***)$





k is the number of rectangles that touch N, ℓ is the number of "active" $\top$ joints.

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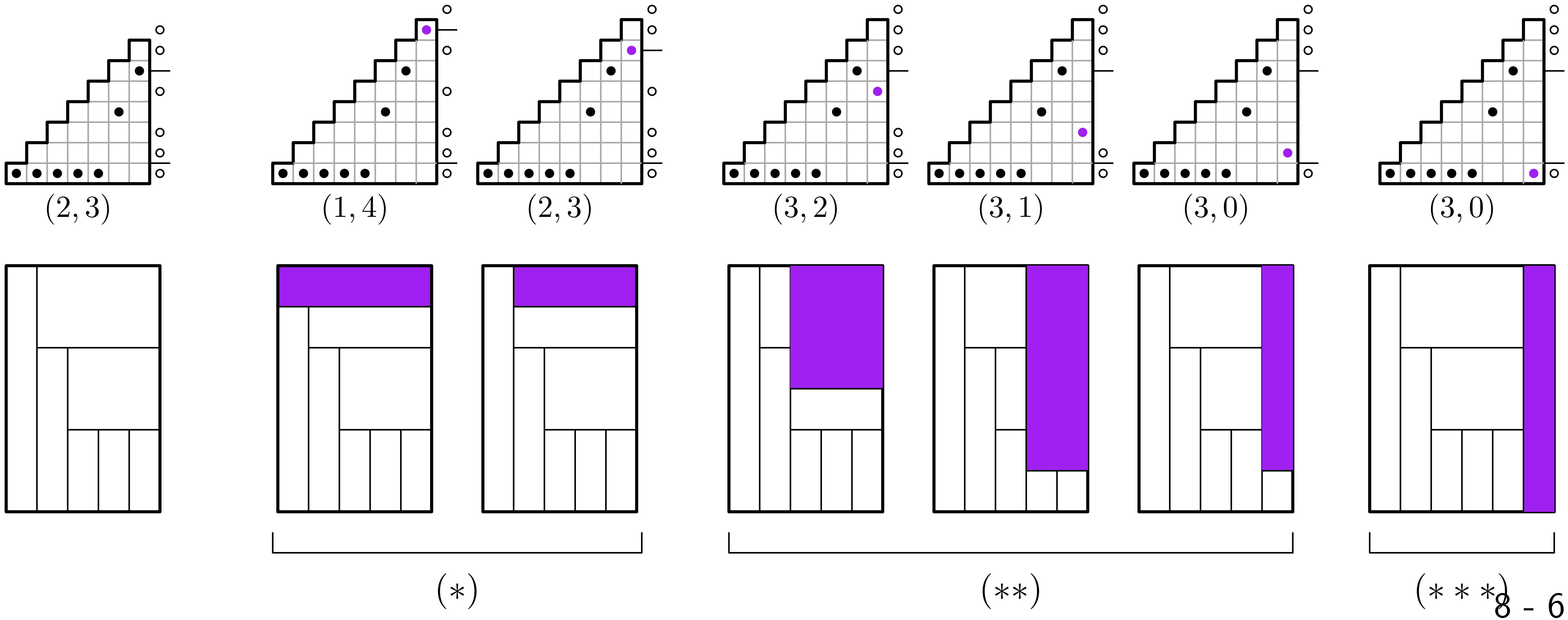
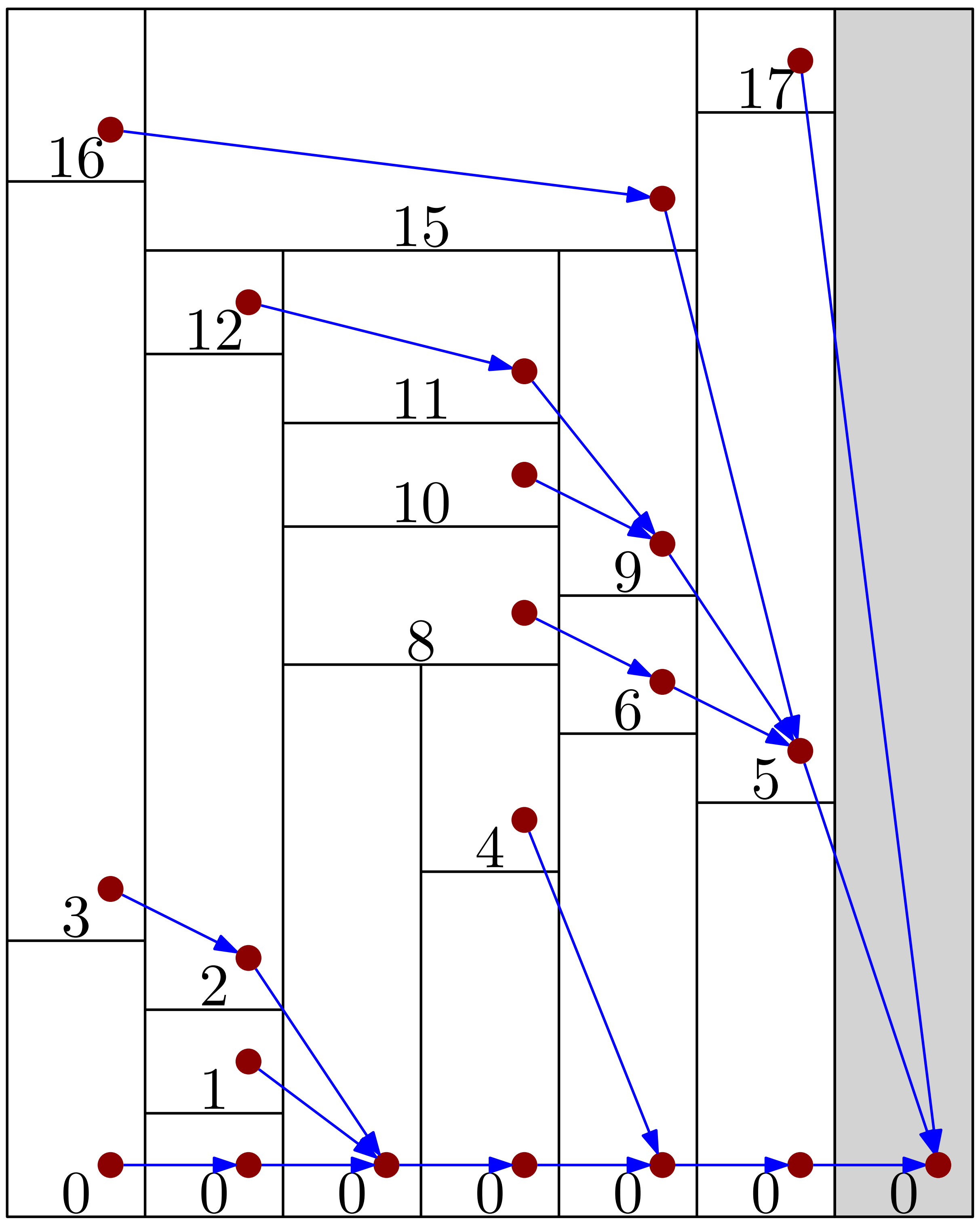
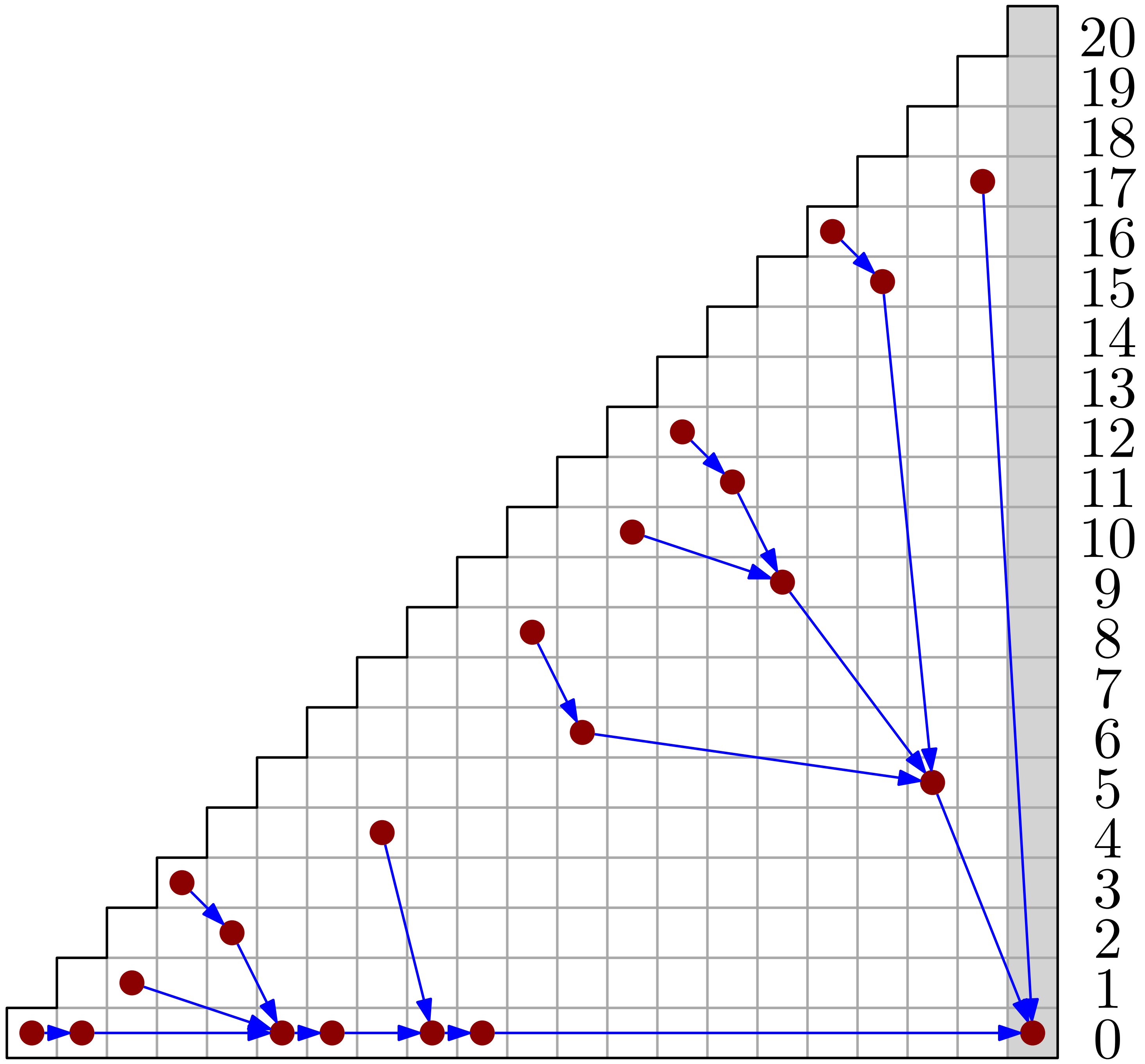


Explicit bijection between $I(011, 201)$ and $\perp$-avoiding rectangulations





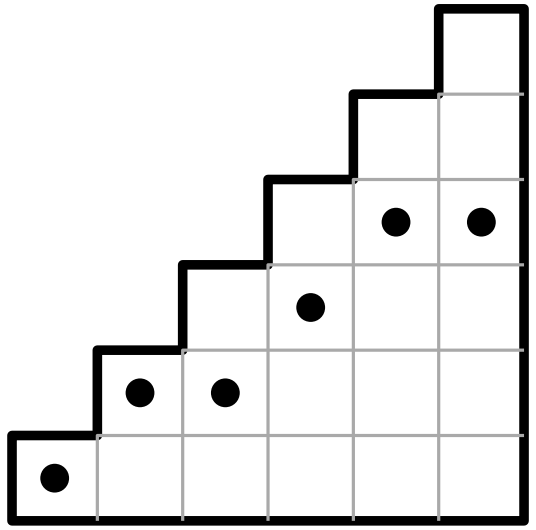
Theorem. For every $n \geq 1$:

1. We have $|I_n(010, 101, 120, 201)| = |I_n(011, 201)|$.
2. The quadruple of statistics (a, b, c, d) for $I_n(010, 101, 120, 201)$, $I_n(010, 110, 120, 210)$, and $I_n(010, 100, 120, 210)$, where

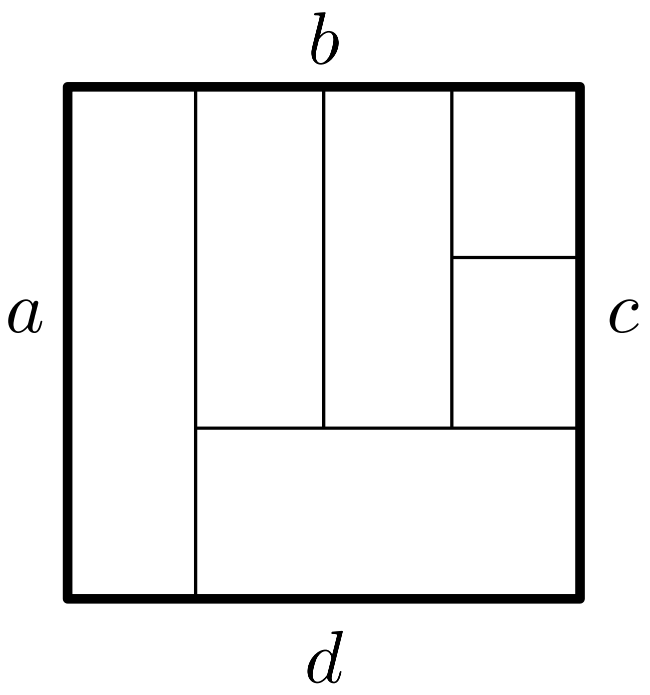
   a is the number of 0 elements, b is the number of left-to-right-maxima,
   c is the bounce, d is the number of high elements.

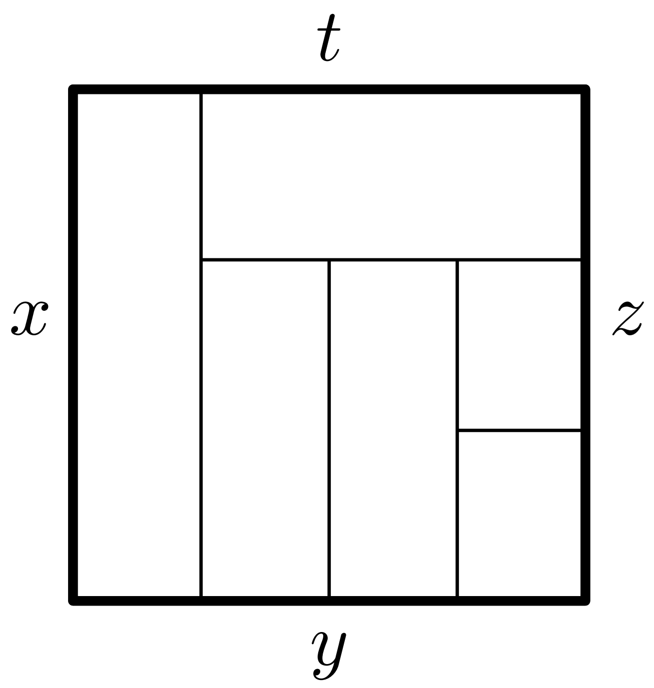
   matches the quadruple of statistics (x, y, z, t) for $I_n(011, 201)$, where

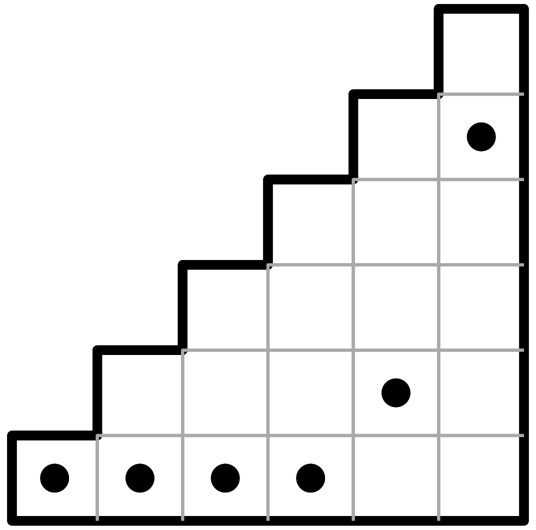
   x is the number of high elements, y is the number of 0 elements,
   z is the number of right-to-left-minima, t is the bounce.



$I_n(010, 101, 120, 201)$





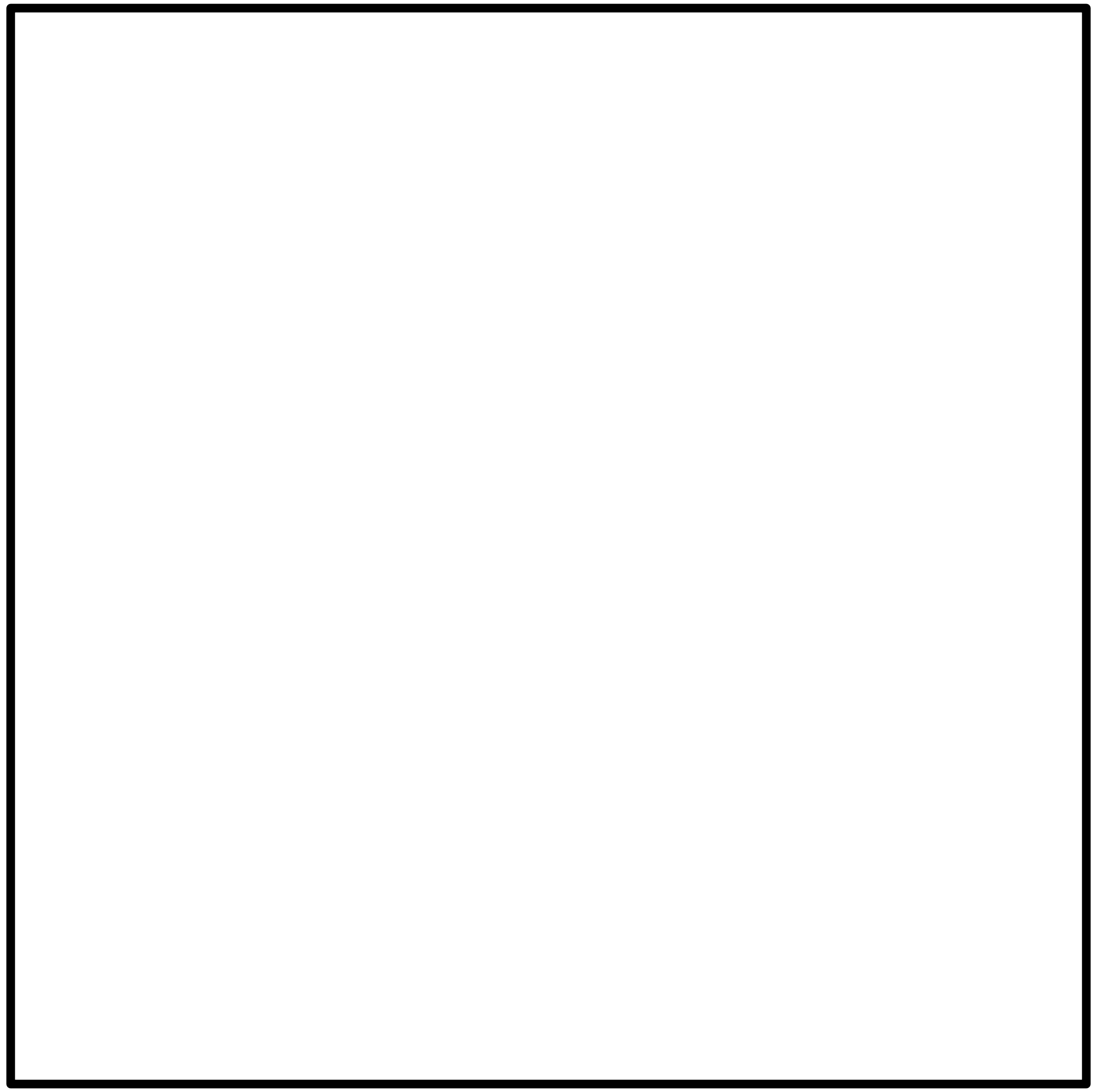


$I_n(011, 201)$

	Weak Equivalence	Strong Equivalence
$\top$	$\|R_n^w(\top)\| = C_n$	OEIS A279555
$\top, \bot$	$\|R_n^w(\top, \bot)\| = 2^{n-1}$	OEIS A287709
$\top, \vdash$	$\|R_n(\top, \vdash)\| = 2^{n-1}$	
$\top, \bot, \vdash$	$\|R_n(\top, \bot, \vdash)\| = n$	
$\top, \bot, \vdash, \dashv$	$\|R_n(\top, \bot, \vdash, \dashv)\| = 2$	

Proposition 3a: $|R_n^w(\vdash, \dashv)| = 2^{n-1}$

Proof: Enumerated by compositions of n.



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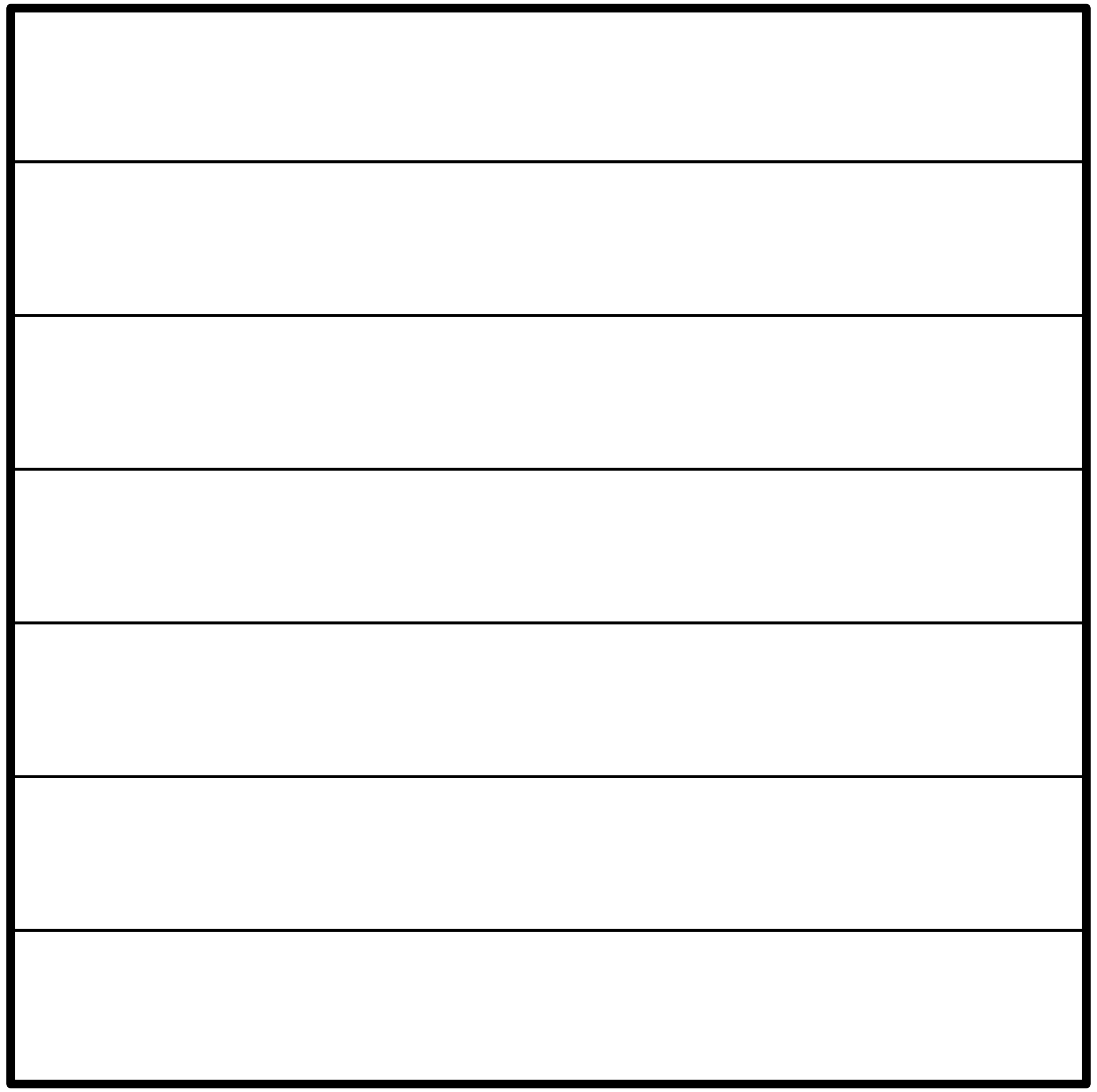

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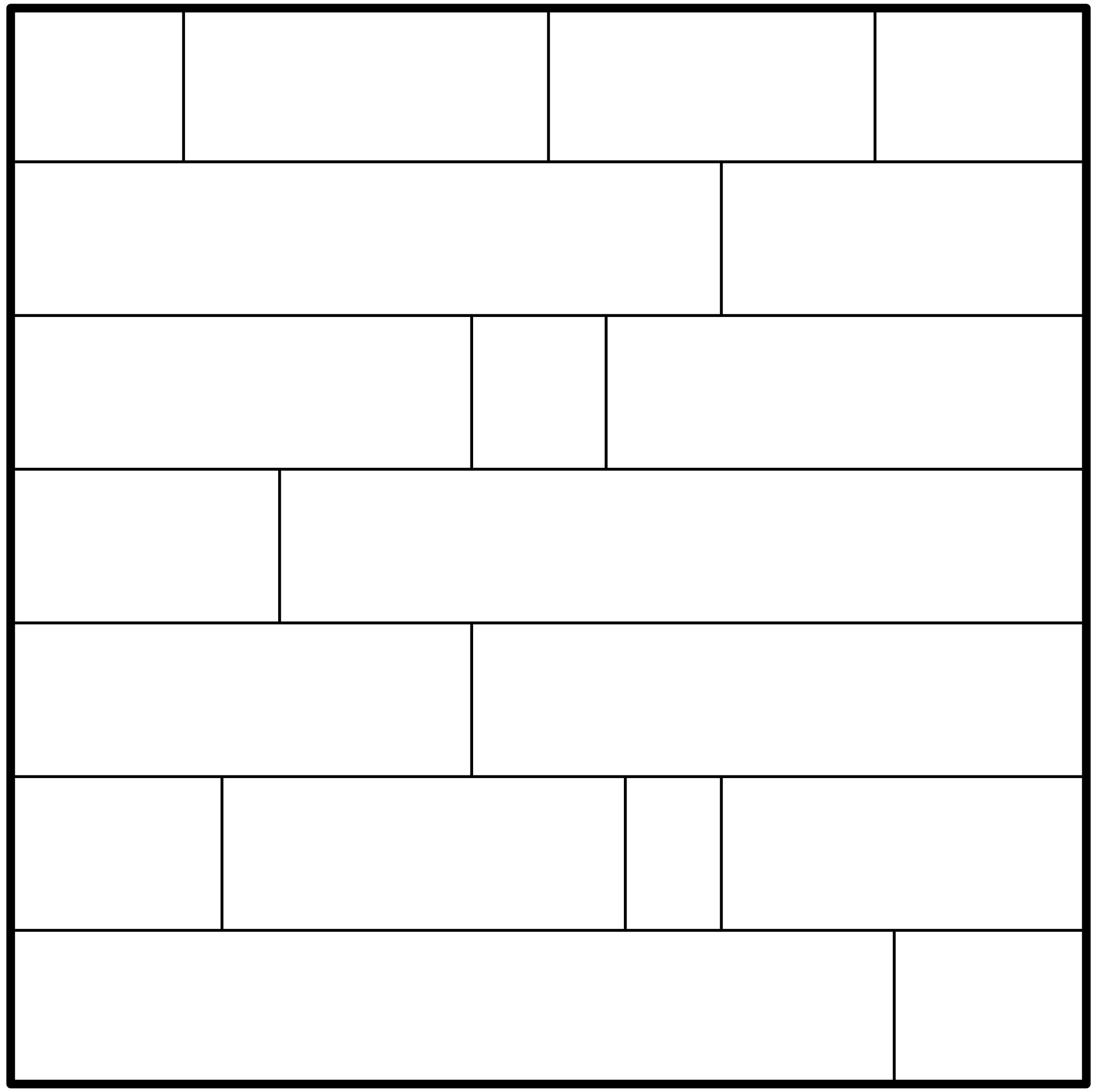

Proposition 3b (Asinowski and Jelínek): Enumerating $R_n^s(\vdash, \dashv)$, OEIS A287709

Proof: Bijection to rushed Dyck paths

A *rushed Dyck path* is one which attains its maximum height on the initial ascent.

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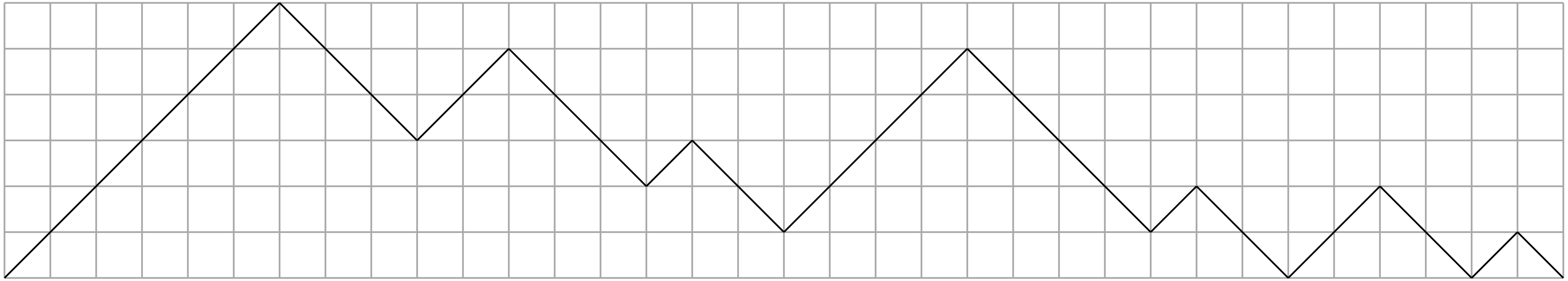

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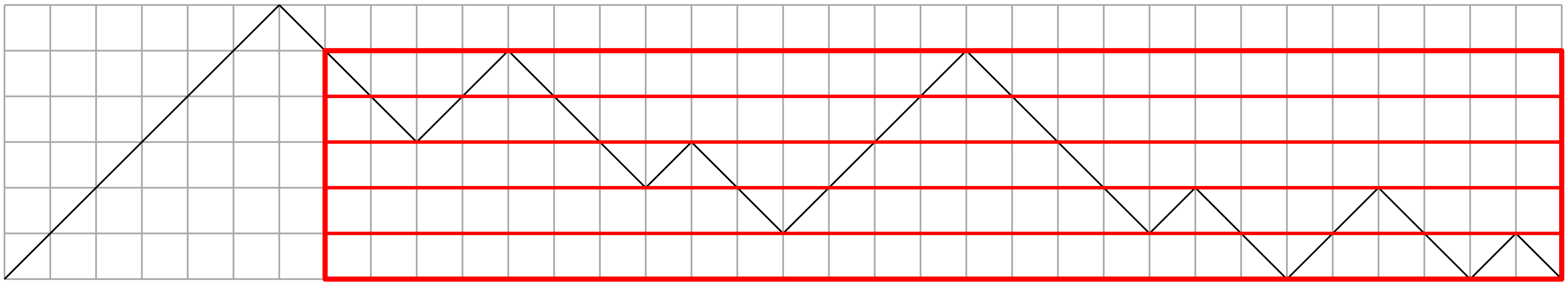

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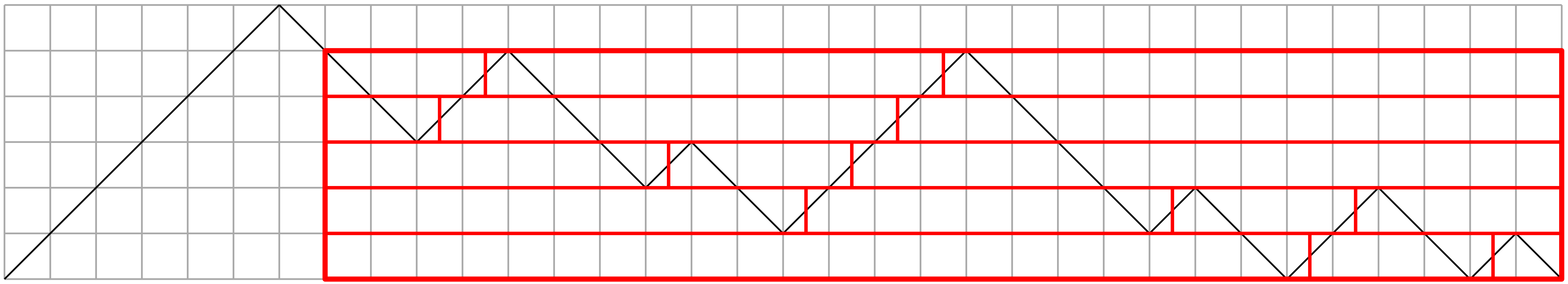

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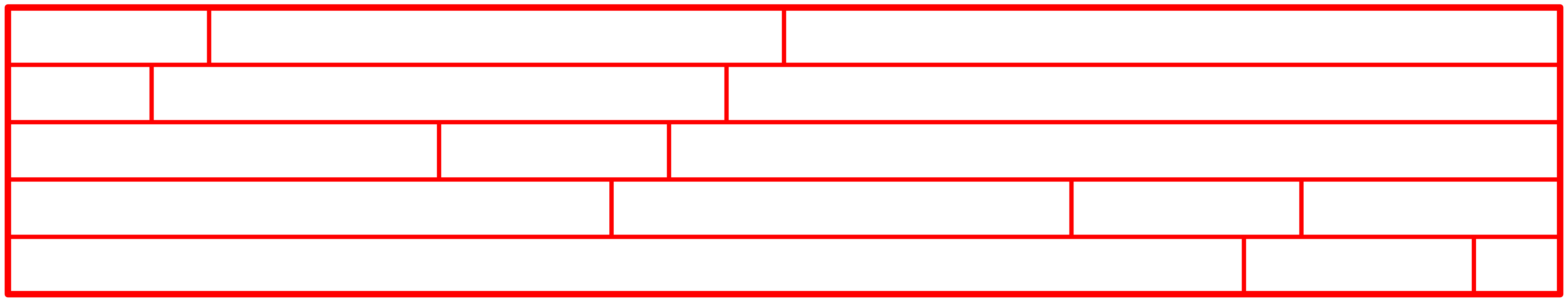

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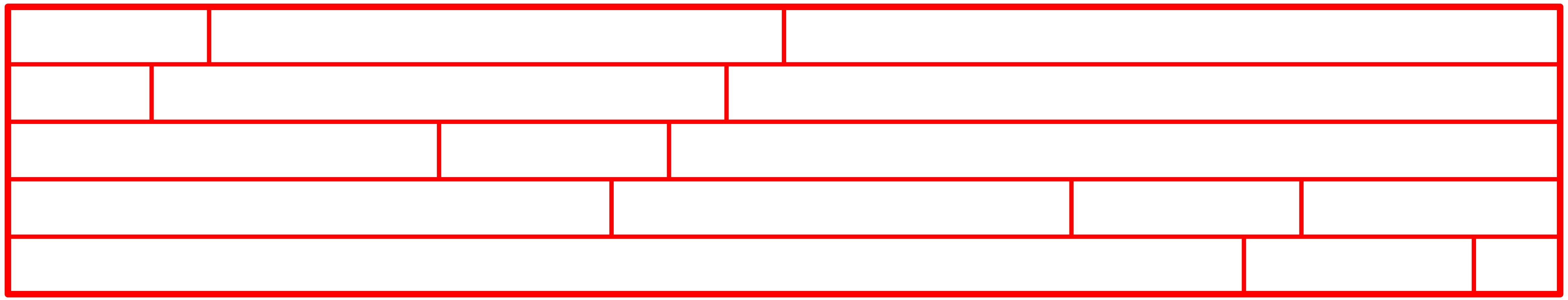
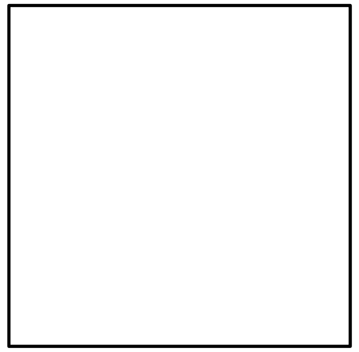
Asymptotics recently proven in a pre-print from Axel Bacher



Proposition 4: $|R_n(\top, \vdash)| = 2^{n-1}$

Proof: Construction of rectangulation



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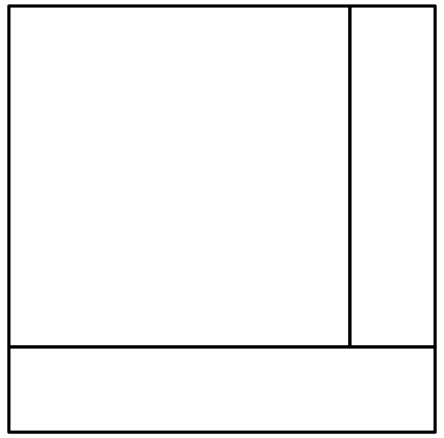

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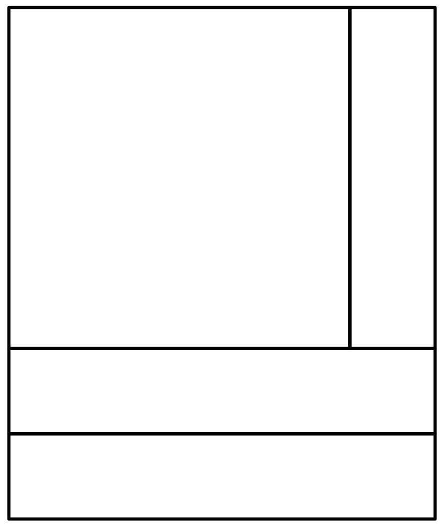

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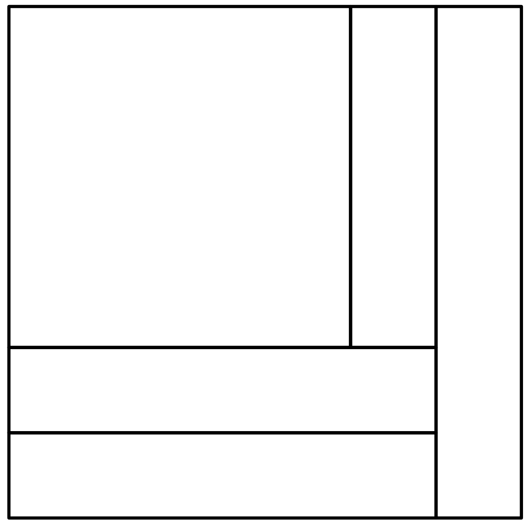

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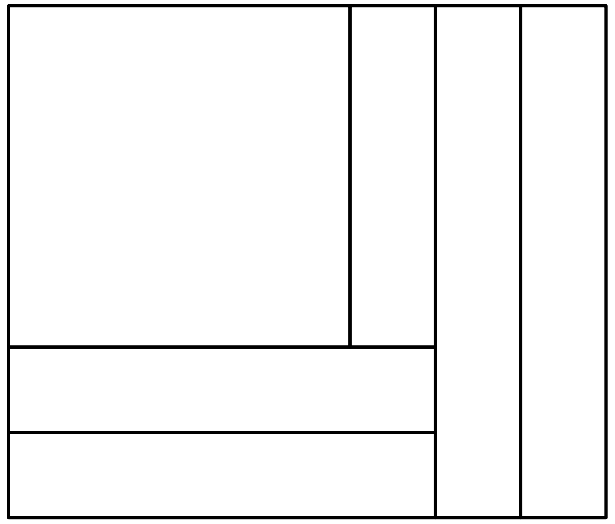

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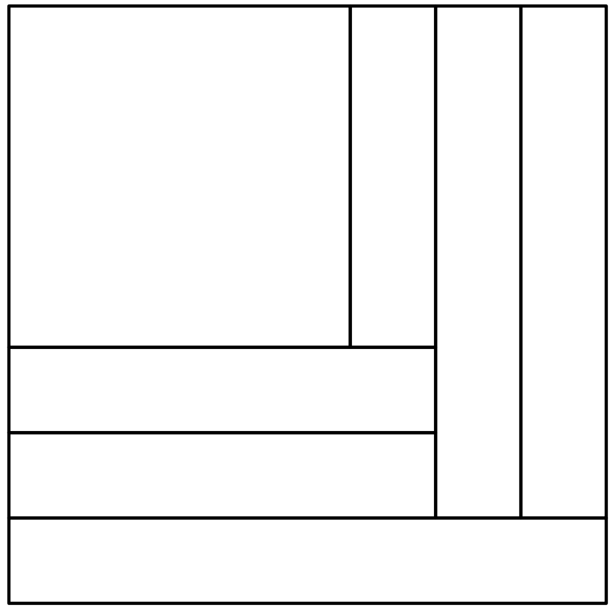

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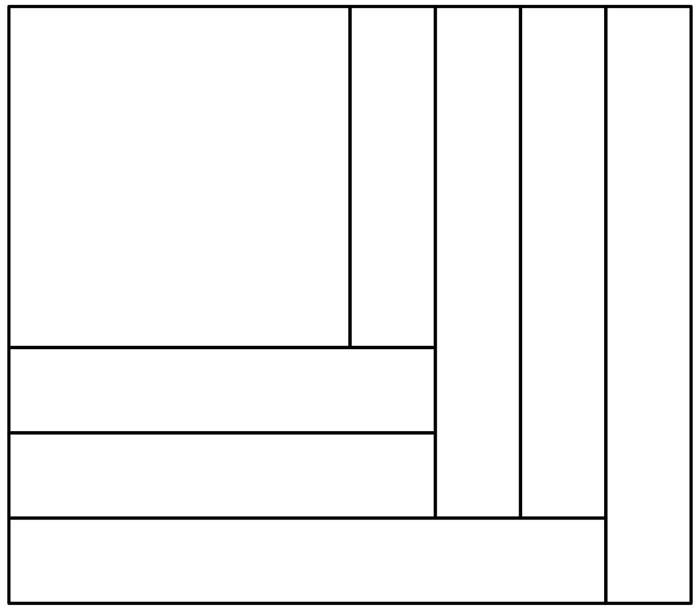

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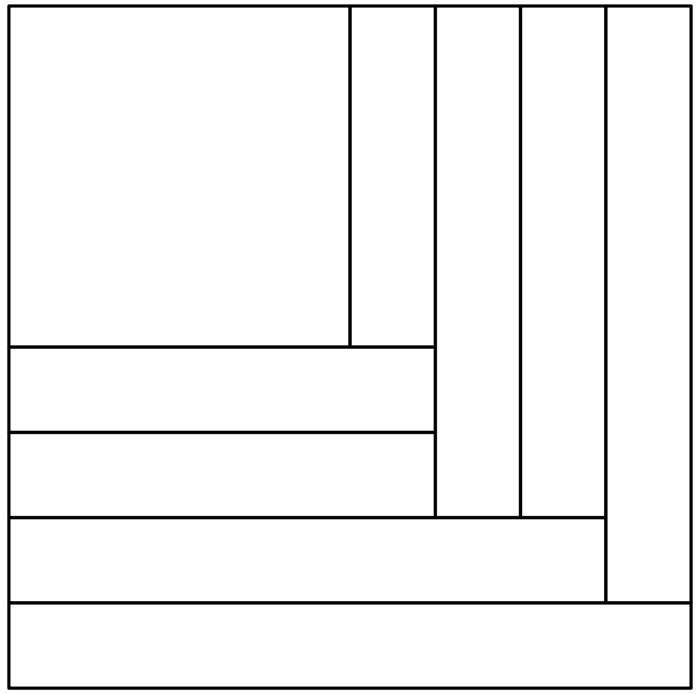
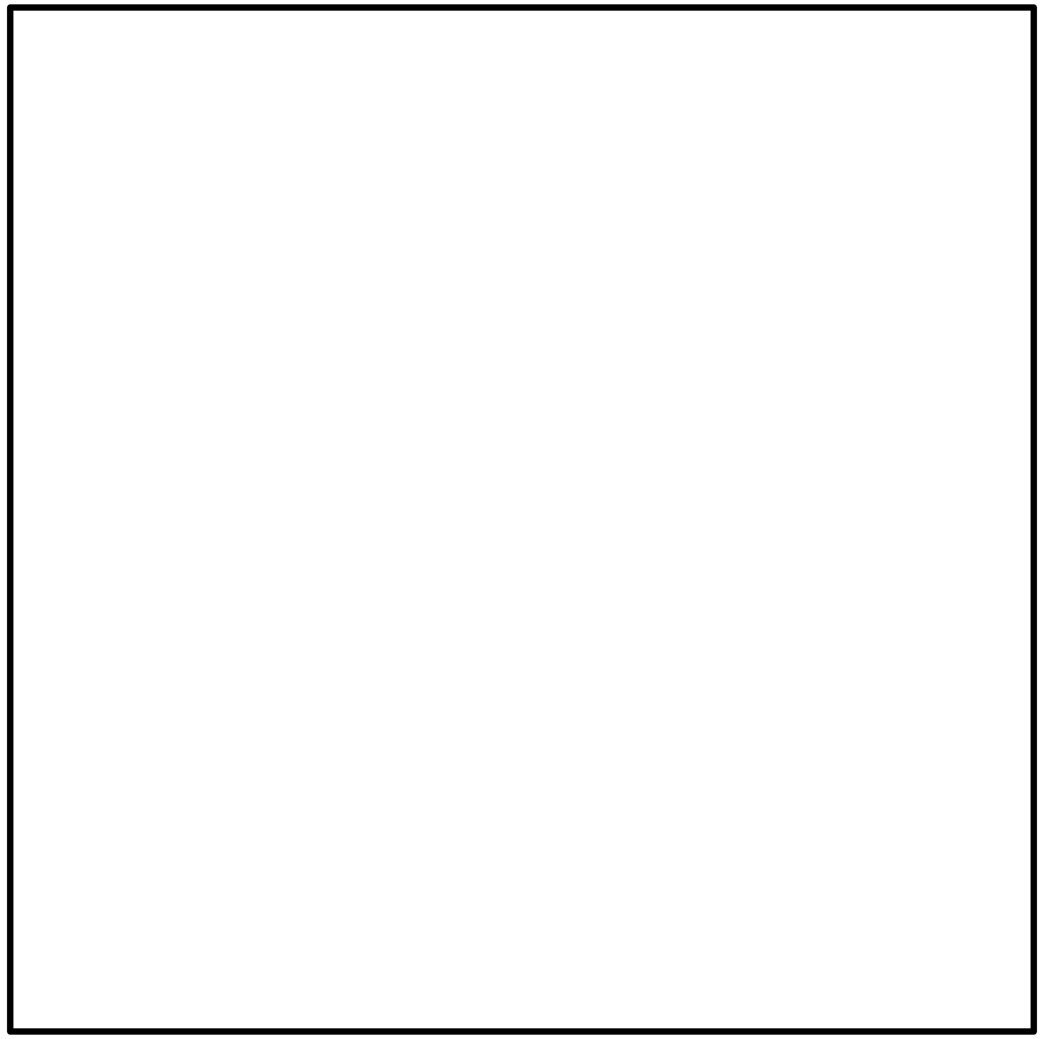


Observation 5: $|R_n(\top, \bot, \vdash)| = n$ and $|R_n(\top, \bot, \vdash, \dashv) = 2$

Proofs: Construction of rectangulations



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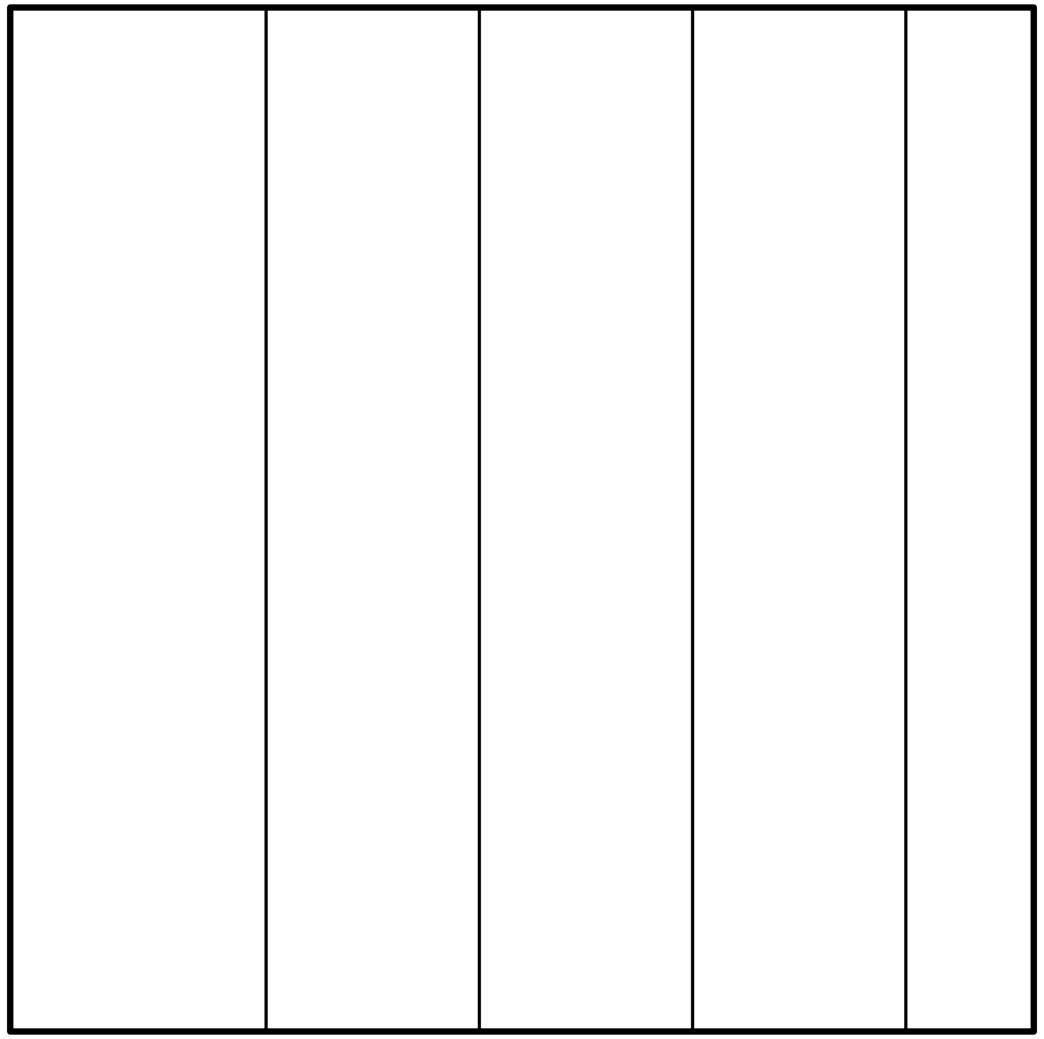

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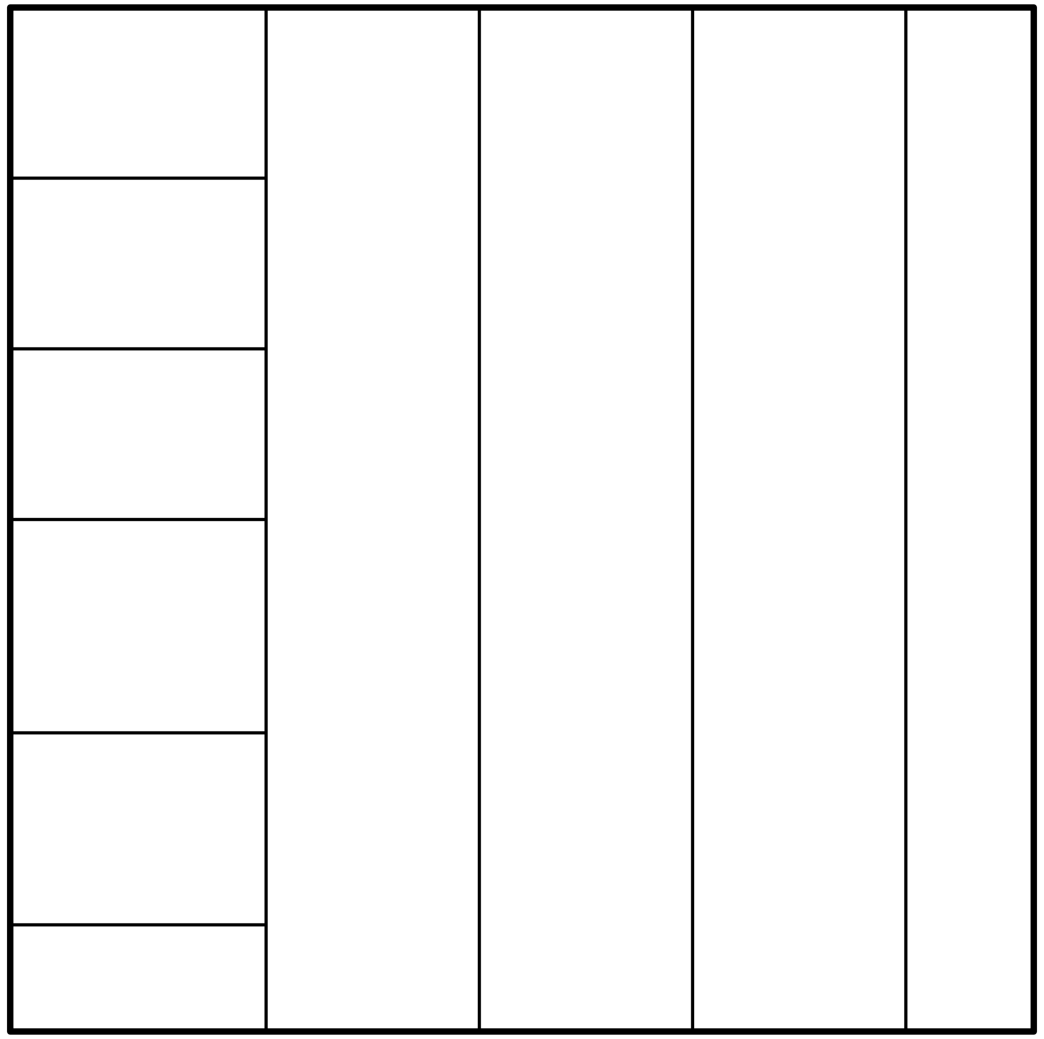

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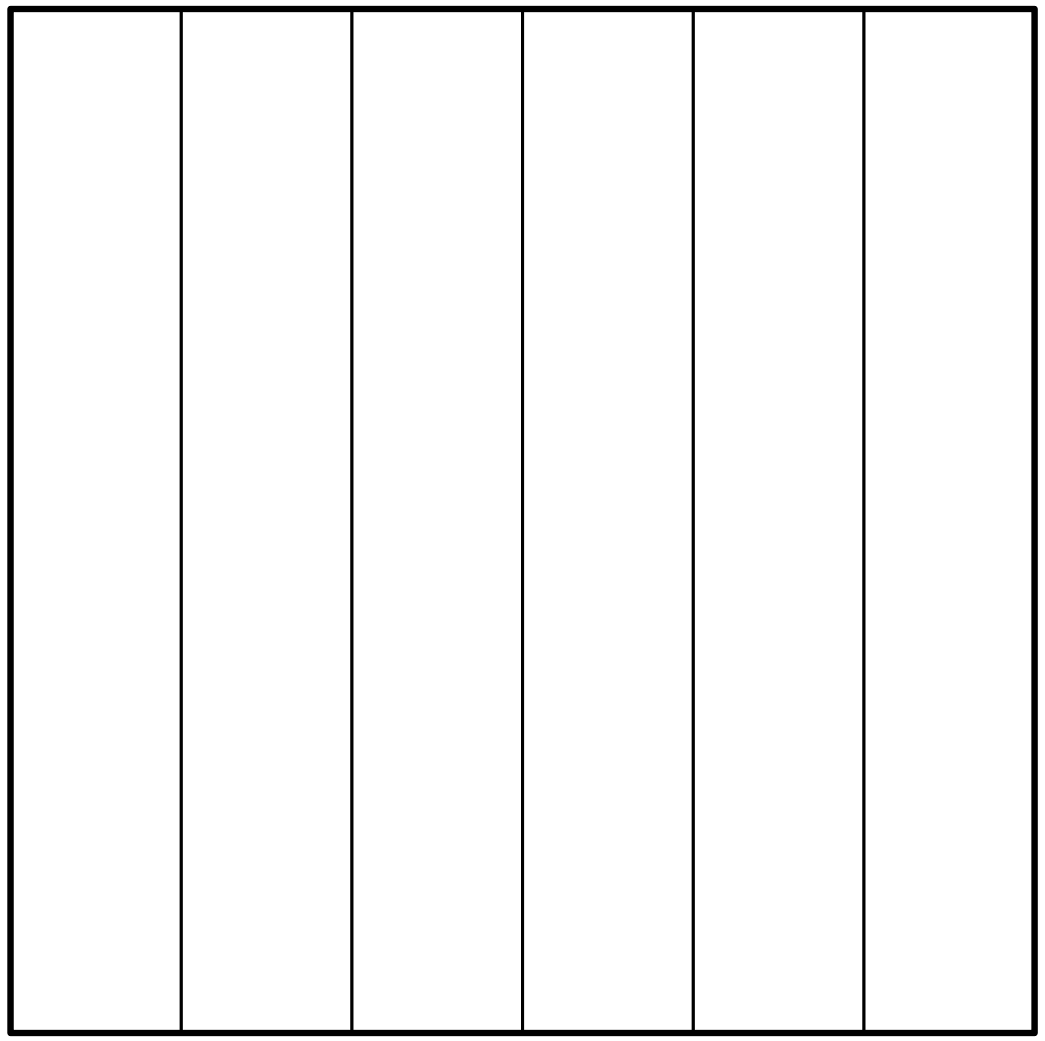

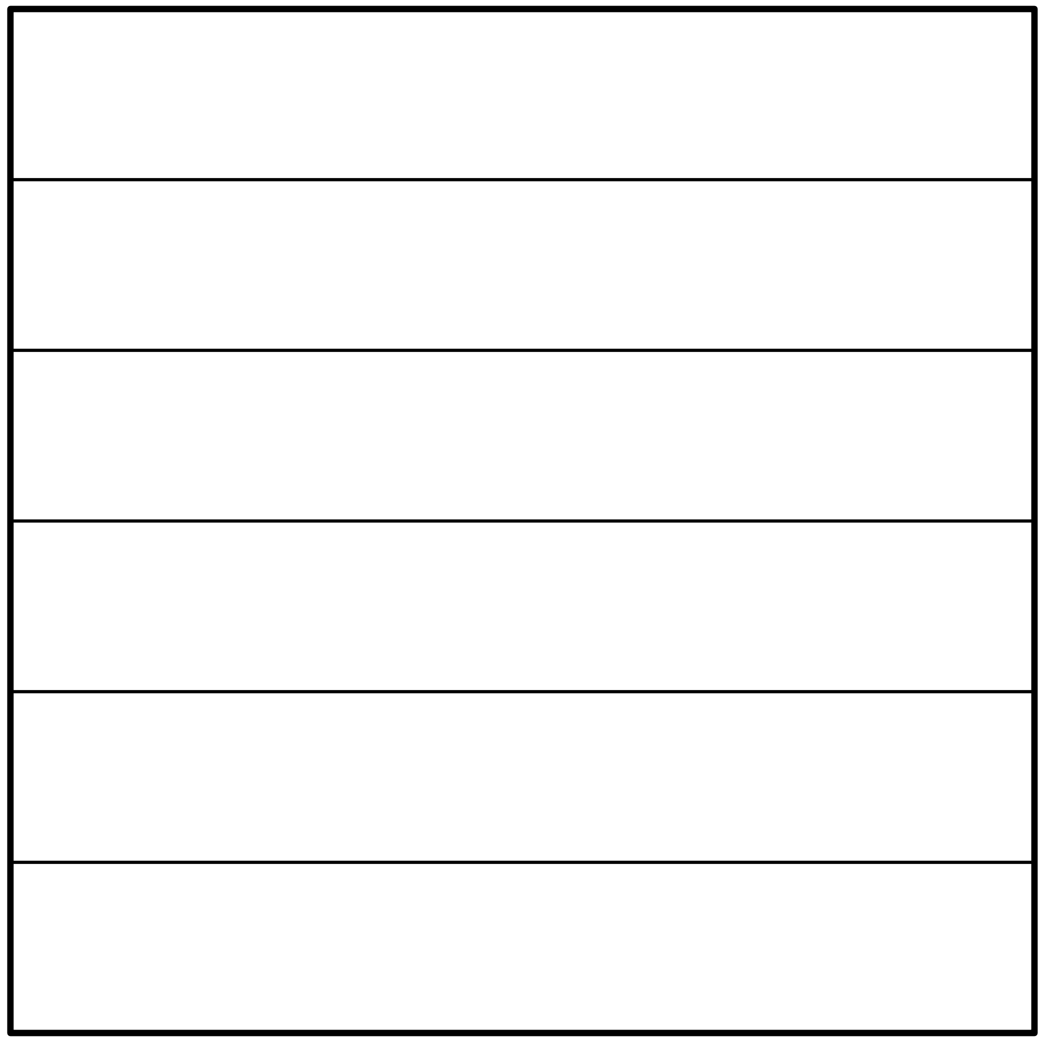

Summary

	Weak Equivalence	Strong Equivalence
$\top$	$\|R_n^w(\top)\| = C_n$	$\|R_n^s(\top)\| = \|I_n(110, 210, 010, 120)\|$
$\top, \bot$	$\|R_n^w(\top, \bot)\| = 2^{n-1}$	Bijection to rushed Dyck paths
$\top, \vdash$	$\|R_n(\top, \vdash)\| = 2^{n-1}$	
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THANK YOU!